CONTINGENT PRICING POLICIES

BY

EUGENE P. DURBIN

TECHNICAL REPORT NO. 76

December 30, 1964

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DEPARTMENT OF STATISTICS

STANFORD UNIVERSITY

STANFORD, CALIFORNIA

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CHAPTER I

MULTIDIMENSIONAL INCENTIVE CONTRACTS IN

DEVELOPMENT PROJECTS

Introduction

An Air Force spokesman succinctly characterized the complexities of
government procurement when he stated:

> ". . . we are supposed to buy at the lowest sound price, on a com-
> petitive basis, but still make sure we support the operational
> force as required, and on time, with quality parts, at the same
> time giving consideration to small business and labor distressed
> areas and without introducing too many nonstandard parts into the
> inventory." [1]

One of the tools employed in the effort to achieve efficient development

of advanced systems is the multiple incentive contract. Although the

Department of Defense (DOD) and the National Aeronautics and Space

Administration (NASA) are not the sole users of contractual incentives,

the bulk of government procurement by dollar volume is performed by DOD

and NASA, and these agencies have pioneered in the use of new contractual

forms designed to increase the effectiveness of the procurement dollar.

Consequently the examples and analysis in this paper will be most perti-

nent to the problems and practices of the defense and aerospace

industries.

The contract types that we will be concerned with are Cost-Plus-

Fixed-Fee (CPFF), Cost-Plus-Incentive-Fee (CPIF), Fixed-Price-Incentive

(FPI), Firm-Fixed-Price (FFP), and Cost-Plus-Award-Fee (CPAF). We

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briefly define the distinguishing features of each contract type. In a CPFF contract the government and contractor negotiate a fixed contractor fee and a target cost. The government is obligated to reimburse the contractor for all allowable costs incurred in executing the contract, whatever the cost outcome. An FFP contract defines the task to be accomplished by the contractor and the total price to be paid by the government for this effort. Any cost variation from this negotiated price is the sole responsibility of the contractor. In both CPIF and FPI contracts a target cost, target fee, and sharing arrangement are negotiated. At the completion of the project, contractor and government together share cost variation according to the negotiated formula. For example, if the sharing formula is 80/20 and the actual cost outcome is $1 million less that target cost, the contractor would receive an additional $200,000, and the government would retain the remaining $800,000. In the CPIF contract the government benefits from all cost underruns below some floor, and is fully responsible for all overruns beyond some ceiling. The FPI contract contains a ceiling beyond which the sharing formula becomes 0/100, and therefore the contractor assumes more risk with an FPI contract than under a CPIF contract. This distinction is more apparent than real since cost outcomes rarely penetrate FPI ceilings. In recent FPI contracts the cost ceilings have been as low as 112-115% of target cost.[1] [4] Almost all contracts with multiple (performance and development schedule) incentives have been CPIF contracts, and DOD policy

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[1]For a more detailed discussion of these contract types see Scherer. [3, pp. 132-142]

has not permitted performance or schedule incentives to be written without cost incentives operating simultaneously.[2]

CPAF contracts are cost reimbursement contracts negotiated with a basic minimum fee considerably lower than CPFF fees. Based on the customer's evaluation of contractor performance an additional award fee of up to 10% can be earned.

Table 1 shows the trend of DOD procurement by contract type from fiscal year 1952 through fiscal year 1964. [5] The increased use of CPFF contracts in the mid-1950's coincided with the accelerated

Table 1

Fiscal Year	CPFF	CPIF/FPI
52	17.9%	12.0%
53	18.0%	26.2%
54	27.0%	27.7%
55	22.7%	24.3%
56	28.4%	21.1%
57	31.2%	19.0%
58	36.4%	22.4%
59	37.7%	18.5%
60	39.4%	16.8%
61	38.9%	14.4%
62	35.1%	16.1%
63	23.4%	27.5%
64	12.0%	32.6%

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[2]We know of one instance in which a completed contract had included the schedule incentive in an initial letter contract. (A letter contract is issued to provide temporary contractual coverage so that urgent work can proceed while a definitive contract is being negotiated.) The urgency of the requirement had justified the use of a letter contract and the buying activity provided the contractor with early motivation necessary to meet emergency programmed requirements. The letter contract provided for both penalty and bonus points applied to target periods. The contractor earned 90% of the incentive reward and the buying activity secured a delivery schedule that had been refused by two other bidders and eight other potential sources. [14]

development of advanced weapons systems and the initiation of major space projects. Between 1955 and 1961 expenditures for development and procurement of strategic aircraft, missiles, and air defense systems averaged approximately $15 billion dollars annually, and during this period the proportion of CPFF awards increased from 22% to 38%. Peck has reported that 12 major CPFF projects completed during this period had an average cost overrun of 320% and an average schedule slippage of 36%. [7, p. 22] Procurement personnel in DOD state that CPFF contracts at best tended to run funds out to the targeted amount, and the term "horror cases" entered the lexicon of defense procurement in reference to some of the CPFF outcomes. [6]

By the end of 1961 the rate of expenditure for the development of new strategic systems stabilized at about $4 billion annually. DOD attention shifted to improving the capabilities of conventional forces, which required less technological development. This allowed the institution of a "Cost Reduction Program" emphasizing both management techniques and contractual innovations. [8] The cost reduction program aimed at conducting development and procurement in an incentive environment, one major element of which was the use of contractual incentives rather than cost reimbursement contracts Unfortunately, in the effort to explain and justify the evolution in contracting, CPFF contracts were blamed for wasting resources, sapping industrial efficiency, dragging out project schedules, and providing no incentive for outstanding performance by the contractor. In fact, it is doubtful whether any contractor could have been induced to accept management responsibility for uncertain, expensive, and technologically advanced systems such as the Atlas missile

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without government assuming the financial risk. Large cost overruns were
in many cases due to customer uncertainty regarding requirements, rather
than poor initial cost estimates. [9] The much criticized time lag caused
by the lack of prior planning and system definition has now been remedied
by explicitly defining the project objectives in a separate contractual
effort that may take up to six months. [10] The flexibility inherent in
the CPFF contract was temporarily ignored in the rush to the incentive
contract.

A revision to the Armed Services Procurement Regulations in 1962
stated the preferential order of contract types to be FFP, FPI, CPIF, and
CPFF. [11] Table 1 indicates the resulting shift to incentives. This
has been explained as an attempt to provide the motivation normally
engendered by a commercial environment, in which competition rather than
a cost outcome determines prices, and competition rather than cost
analyses insures that profit remains fair.[3] As the complexity and expense
of weapons and space systems have increased, their useful life has
decreased. There is now little opportunity to retrofit or modify an
operational system. Consequently, early design attention in the areas
of maintainability, reliability, and standardization can achieve large
"downstream" savings. Systems are also now procured in smaller quantities,
and development funds constitute a larger fraction of total lifetime
costs. Thus while superior operational performance is desirable, it must
be balanced against development cost.

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[3]Under conditions of price competition the government may award FFP
contracts to the lowest bidder without regard for profit. [12]

The use of cost incentives was relatively straightforward, but
multiple incentives raised a host of questions regarding the definition
and measurement of performance elements, the determination of the relative
value of alternative outcomes, and the formulation of fee patterns to
motivate contractor decisions consistent with government objectives.[4]
In the past year techniques have been proposed that lead to fee structures
more compatible with estimated government values. [15, 16] However, an
inherent weakness of contractual incentives is that the ranges and weights
assigned to performance variables, costs, and profits are negotiated early
in the development program, while the contractor capability and potential
operational value vary throughout the development effort.

Cost-Plus-Award-Fee contracts have obvious merit in advanced develop-
ment and have been used most extensively by NASA and the Navy. They
retain the flexibility of CPFF contracts while providing even more

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[4] One of the first multiple incentive contracts went to TRW Space
Technology Laboratories. Ten nuclear detection satellites were to be
constructed at a basic cost of $14 million, and a basic fee of $1 million.
A successful launch required that two satellites be orbited simultaneously.
Incentives were paid depending on the number of attempts required to
achieve a successful launch.

First Success at Launch Number	Award Fee
1	$125,000
2	$ 98,000
3	$ 0
4	-$ 98,000
5	-$125,000

The expected lifetime in orbit was 2-4 months. For every day short of two
months in orbit STL was penalized $1600, while for every day in orbit from
four to six months, STL received an additional $1600. It would be diffi-
cult to objectively explain why these fees were chosen, except to note
that the $250,000 launch fee swing is 25% of the basic fee, and that the
maximum fee based on operational lifetime is approximately $100,000 per
launch, or 10% of the basic fee. [13]

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incentive than the possibility of increased profit. The very fact that profit is based on evaluation of contractor performance causes the contractor to perceive a relation between the current and potential future contracts. This appeal to the incentive for organizational survival may far outweigh any incremental profit.

An additional consideration in attempting to increase the effectiveness of procurement policy is the increasing scope of mathematical programming models. Zschau [17] has demonstrated that under reasonable conditions a large development project can be decomposed into subprojects, and this decomposition used to obtain minimum cost surfaces for every feasible schedule and performance outcome of the total project. This suggests that a similar approach be used to periodically evaluate the worth of alternative system performance as development progresses. Before developing this idea further we will discuss the behavioral assumptions and information required to use contractual incentives.

Contractor Motivation

Contractual incentives are based on the assumptions that the contractor can exert a known degree of control over the product or system, and that he will act to improve his perceived corporate position. Economists completely characterize contractor behavior by stating that program decisions are made to maximize utility, but this is not an operationally useful statement in structuring an incentive fee.

Classical models of the firm assume that production and pricing decisions are made to maximize some form of monetary profit, which may be current profit, discounted future profit, or expected profit per period. In a cost incentive development situation the firm "produces"

7

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final cost. The fee is usually a constant plus some fraction of the difference between initial target cost and final cost. If final cost is controllable by the contractor, and he maximizes current fee, he will choose the minimum possible final cost, regardless of the sharing proportion. The fact that cost targets are rarely underrun by large factors suggests that cost reduction actions cause disutility. Scherer has explained this with a user cost model which we describe briefly. [3, p. 407]

User cost is defined to be the future profit loss resulting from current cost reduction actions. A contractor may initially effect cost reductions by increasing efficiency and reducing unnecessary waste, but beyond some point further cost reductions can be achieved only by such actions as laying off personnel, breaking up design teams, or curtailing effort in development. The contractor perceives this as producing a less attractive system in the long run, thereby adversely affecting his likelihood of receiving future contracts. Assume that the user cost can be represented as a quadratic function of the cost underrrun.

Let C_t = Target Cost

C_a = Actual Cost Outcome

$X = C_t - C_a$, Amount of Cost Underrun.

Then $U(X) = a + bX + cX^2$, and we see that depending on the coefficients, a, b, c, cost underruns may lead to high user costs. Assume that the contractor maximizes the difference between accounting fee and user cost through his choice of X.

Let π_a = Accounting Fee

π_t = Target Fee

α = Sharing Proportion

Then $\pi_a = \pi_t + \alpha X$, and the condition that X^* maximize $\pi_a - U(X)$ is $\frac{d}{d\alpha}[\pi_a - U(X)] = 0$, or $X^* = \frac{\alpha - b}{2c}$. Given α, and the user cost function, an underrun will not always be optimal. The contractor's problem in negotiation is to choose the α that maximizes the final net profit, $\pi_t + \alpha X^* - U(X^*)$. Scherer shows that except in special cases, the optimal α polarizes to 0 or 1, that is either CPFF or FFP coverage. Since the average value of α observed in Scherer's case studies was 0.2, he tested the hypothesis that contractors maximize profit subject to a constraint on maximum profit. This behavior would be explained by the attempt to avoid unfavorable publicity and interest on the part of the Renegotiation Board and the General Accounting Office. The following case described by the president of Giannini Controls Corporation is an instance of the operation of these factors.

> Chart No. 2 (not shown) shows the cumulative margin earned in one of our divisions on a series of contracts for heavy electronics equipment. We started at a heavy loss in 1960 because all startup costs and some developmental engineering applied against that first contract, but as production increased we showed a satisfactory trend. Obviously we made profit on every contract after the first one, since the cumulative margin rises, but that startup hole was pretty deep and we didn't break even until early 1962. During this time prices were reduced substantially, our quality record was excellent, and we were shipping equipment ahead of schedule. In every way we were rated an exemplary supplier. Looks pretty typical, doesn't it? We invested heavily in an attractive program, covered our first costs, brought the cumulative profits up to industry average, and satisfied our customer completely. And then we ran into the buzz saw. The prime agency audited a late 1962 contract that had somehow earned an unusual profit: over 20%. We had never made close to that on any prior or subsequent contract, and the order at issue was only $200,000 out of $4,000,000 total business. Our margin before that windfall was just over 4%, and with the

"excessive profits" included we barely reached 6.4%, but the contract
stood alone and none of this background carried any weight. The
result was exactly 11 months of transcontinental debate with the
agency demanding a profit rebate. We wouldn't agree to refunding
a prior year's earnings, but we did accept a downward redetermination
of the current production price. That's why you see the margin
slipping off a quarter point in 1964--we're paying a penalty now for
conservative estimating on a single contract in 1962! [19]

If such occurrences were frequent the weight assigned to profit as a

motivator of contractor behavior would be low. Peck concluded, however,

that risk aversion rather than concern over renegotiation motivates the

low sharing proportions observed, and Scherer also stated:

Much more frequently, contractor representatives mentioned risk
aversion as the principal reason for their efforts to negotiate low
values of α. They stated explicitly that in many instances, given
the uncertainties which pervaded advanced weapons system and sub-
system production programs, they were willing and eager to sacrifice
the higher average profit expectations associated with firm fixed
price contractual coverage for the greater security against an
occasional short run loss afforded by CPFF and incentive contracts.
[18, p. 275]

Baumol suggests that many large commercial firms make pricing,

advertising, and marketing decisions with the objective of maximizing

total sales revenue, subject to a constraint on the minimum profit that

will satisfy stockholders. [20] This hypothesis is only partially

applicable to the large defense contractors, whose sales are measured by

the dollar volume of research, development, and production contracts

held. Too rapid an expansion of this type effort results in a dilution

of product quality, leading to a poor corporate reputation. Consequently,

defense contractors prefer to expand gradually, keeping key research and

design teams intact and productively employed.

There are less quantifiable motivators pertinent in predicting

corporate behavior. These include organizational survival, security of

employment, sales, and profit, freedom from harassment, the desire for

public approbation, the desire to enhance national security, and the desire to advance science and technology. [3, p. 7] The research oriented contractor is especially interested in retaining his key scientists and maintaining the continuity of his research effort. Thus while contractor negotiators may act as if they intend to maximize current accounting profit, there is evidence that program decisions are based on the longer range factors. [3, p. 159] In some cases the advantages accruing from pioneering in a new technical area may be sufficiently strong to motivate a cost-plus-no-fee contract, and in all cases the profit motivation must be related to both the project uncertainties and the intangible contractor rewards. In the current atmosphere of limited and selective development, sharing proportions are more often 0.4 or 0.5 than 0.2, and the revised profit policy of DOD emphasizes profit as a stimulant to contractor effort. Fewer projects are being carried into engineering development, and projects in research, exploratory, and advanced development are subject to immediate cancellation if more promising concepts appear. [23, 24] Under these conditions concern for organizational survival is very real, and there is less of a requirement for extensive contractual safeguards and stimulants.

Scherer's user cost model is descriptive in explaining contractor behavior in cost reimbursement and cost incentive contracts. We would proceed one level deeper in an effort to understand the cost-time-performance choices made by the contractor. We hypothesize that the defense contractor acts to maximize the probability of program continuation subject to a minimum profit constraint. A single very large loss, or a succession of unprofitable projects will drive down the market

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price of the firm's securities, adversely affect the firm's image among the agencies awarding contracts, and raise doubt as to the capability of the firm's management. Hence the program manager is constrained in the short run against outright loss, and in the long run to some minimum profit. During any given development contract, we feel that the program manager and other corporate personnel have a perception of the customer valuation of the different parameters of the system.[5] Adjustments to these perceived values are obtained from the in-plant agency representatives, visiting service personnel, liaison personnel in Washington and other headquarters, and from knowledge of the progress of potentially competitive projects. The program manager can therefore judge the relative desirability of alternative schedule, performance, and cost outcomes from the buyer's point of view. His initial system choices are determined by the acceptable time-performance outcomes defined by the development contract, modified by the current view of the buying agency, and constrained by the cost-time-performance surface. If the cost resulting from a desirable outcome is less than target cost, the manager decides whether to take the cost underrun as a saving for the buyer, or

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[5] Our survey of aerospace contractors revealed one development project incorporating an incentive for the successful firing of a missile by a certain date. Failure to successfully fire by that target date would result in successively reduced incentive payments, until at the end of six months a substantial penalty would be levied against the contractor. The buying agency was aware that pressure existed to halt program funds and realized that an early successful firing was necessary. The contractor also recognized that failure to demonstrate the system successfully would increase the likelihood of program cancellation, and therefore put forth a "crash" effort. It is our opinion that the pressure for program continuation was more important than the contractual reward at stake.

to utilize it to increase performance in profitable areas. The latter decision enhances system attractiveness relative to potential competitors. If, however, the characteristics considered essential by the customer are only attainable at a cost above target, the program manager must decide whether performance or monetary considerations are more important in influencing program continuation.

At any program decision point the contractor behavior is then conceptually described by a nonlinear programming problem in which the firm chooses performance variables, schedule outcomes, and development cost to maximize the probability of program continuation subject to a constraint on minimum profit and contractual constraints on the values of performance, schedule, and development cost.

Let c = cost outcome aimed at by the contractor,

t = schedule outcome aimed at by the contractor,

p = vector of performance outcomes aimed at by the contractor, $p = (p_1, \ldots, p_n)$,

$\bar{c}$ = current cost ceiling desired by the buyer, as perceived by the contractor,

$\bar{t}$ = current development schedule desired by the buyer, as perceived by the contractor,

$\bar{p}$ = vector of performance outcomes desired by the buyer, as perceived by the contractor, $\bar{p} = (\bar{p}_1, \ldots, \bar{p}_n)$.

Let $\psi(c,p,t) = 0$ represent the minimum cost attainable at every time and performance outcome, and let $\varphi(c,p,t)$ denote the contractor fee at any outcome. Assume that the probability of program continuation approaches one as the projected project outcome approaches the outcome

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desired by the customer. Assume further that the program manager estimates the weights β_i, and that minimizing

$$\sum_{i=1}^{n} \beta_i(\overline{p}_i - p_i) + \beta_{n+1}(c - \overline{c}) + \beta_{n+2}(t - \overline{t})$$

approximates maximizing the probability of program continuation. The contractor problem is then

$$\underset{c,p,t}{\text{Minimize}} \quad \sum_{1}^{n} \beta_i(\overline{p}_i - p_i) + \beta_{n+1}(c - \overline{c}) + \beta_{n+2}(t - \overline{t})$$

such that

$$\psi(c,p,t) \leq 0$$

and

$$\varphi(c,p,t) - \psi(c,p,t) \geq 0$$

In view of the information and decision variables entering the contractor decisions under this hypothesis, we conclude that expected fee alone is a weak and uncertain motivator. Even if the cost-time-performance tradeoff surface were stationary and available to the buying agency, and if a set of multiple incentives could be structured consistent with true government values at every outcome and assuring maximum contractor fee at that outcome desired by the government, there is no assurance that

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this outcome would be chosen by the contractor. Contractual incentives can be analyzed for consistency, but not for motivation.[6]

We conclude that in research and exploratory and advanced development, the primary contractor motivation is to maintain the firm's capability to compete for and retain projects. There is increasing realization that only a few systems will proceed into engineering development, and fewer still into production. A greater amount of effort will be directed into research and exploratory development, and profit will be based on the cost of the research effort as well as results. These conditions create incentives for efficiency, and the marginal value yielded by the superposition of contractual incentives may not be worth the effort required of the contractor and procurement personnel.

Government Objectives in Development

Regardless of the source of the requirement for a new weapons or space system, a preliminary study of projected system utilization is necessary to specify initial performance characteristics and operating conditions. In structuring an incentive contract the government must have a ranking of the possible development outcomes, and even in a CPFF contract where decisions on specific details may be made later in the program, recommendations should be based on some consistent evaluation of

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[6]This model of contractor behavior is consistent with Peck's observation that firms are more prone to take risks if their projects depend on it. [7, p. 540] For example, if the purchasing agency is debating program continuation and has suspended funds, the contractor may supply corporation funds to continue critical development of production until a final decision is reached. Not to do so might affect the final decision unfavorably, while in the event of program termination some of the expended funds may be recovered.

system worth. Yet this task of determining operational value is extremely difficult and controversial. The Holifield Report criticizing Defense management of satellite communications is quoted as stating:

> "The unfortunate experience (of DOD analysis of alternative satellite communications systems) suggests to us," the report continues, "that considerations of economy from a Defense Department budget stand-point are not sufficient criteria for making decisions in fields which involve government-wide policy, politics, and international diplomacy." [25]

Sidestepping the question of whether a particular system should have been brought to the development stage, the buyer may select various objectives in the development effort. The more obvious are

    (i)   maximizing some measure of effectiveness subject to a cost constraint, and

    (ii) minimizing the lifetime cost of the system subject to the attainment of some fixed level of effectiveness.

There may be other valid objectives such as minimizing the number of military personnel required to achieve a given level of effectiveness, or obtaining a system flexible enough to operate in a number of possible environments. In any event development objectives and government negotiation objectives are not identical. The former are determined by evaluation boards, cost-effectiveness analysts, and military personnel, while the contract negotiations are performed by procurement specialists beset by a welter of regulations, guidelines, and hindsight. Hence the Director for Procurement Policy, OSD, testified:

> A requirement (for a weapons system) having been established, and feasibility determined, what is our objective? It is to procure the development, production, and delivery of the weapons system into inventory in the shortest possible time and at the minimum cost. [27, p. 106]

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But Scherer's case studies indicated that actual government procurement practice has been a compromise between maximizing the incentive for contractor efficiency, minimizing contract outlay, and minimizing the risk that unnecessary and excessive profits will be paid.[7] [3, p. 147]

There is an effort in current procurement policy to consider total system costs and not be hesitant about spending funds in development to achieve operational savings, but at the initiation of development the potential value of a new system is extremely uncertain. The cost-effectiveness analyst assumes that development will be successful and tries to foresee the implications of the successful system in the relevant time frame. The essentiality of specific characteristics will vary with developments in associated systems, in opposing systems, and in the operational environment.[8] Preliminary models of the system attempt to determine the key parameters, and the variation in performance as a function of these parameters. After preliminary studies have shown a concept to have merit, the Project Definition Phase (PDP) may begin. PDP is a formal step in which the cost-effectiveness model is refined

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[7]Minimizing contract outlay is especially shortsighted in view of the operating and maintenance expenses of modern weapon systems. In an expository example Hitch and McKean point out that a hypothetical air transport fleet obtained under the minimum contract outlay would cost over \$1 billion more than the least cost fleet on a lifetime basis. [26, p. 140]

[8]An appropriate approach in this situation would seem to be a sequential decision process, with specification details supplied to the contractor as late as possible to take advantage of available information. According to one theory Defense programs may not be managed in this manner because the members of development staffs recognize the transitory nature of their assignments and hesitate to leave vital decisions to their unknown successors. Hence the system is "cast in concrete" at an early point. [28]

further, and the total system costs and development schedule estimates are determined. It would be desirable to have government personnel determine the relative value of the possible outcomes, but normally two or more contractors will conduct PDP, competing for the follow-on work. During PDP these contractors will determine the "optimum balance between total cost, schedule and operational effectiveness for the system." [29] Following PDP a development contract may be awarded on an FFP or FPI basis, using the tradeoffs generated in PDP as a basis for the incentive structure.

Consistent incentives require the capability to compare performance increments to cost increments. Therefore the analyst must find some method of ultimately comparing different aspects of performance in dollar terms. This may be straightforward in some cases. For example the value of early delivery of a military satellite communications system might be indexed by the current dollar usage rate of commercial facilities. The problem is more usually akin to determining the worth of additional "rate of climb" in an anti-submarine aircraft. Quantitative statements about the worth of accelerated development undoubtedly include consideration of possible system obsolesence due to oncoming competitive systems, or the need of the agency to demonstrate a successful prototype to obtain further development funds. Development programs are basically either urgent, and rapid development receives priority, or else the purchasing agency is interested in showing cost efficiencies, and only enough is spent to keep the schedule from making uneconomic use of resources. Scherer concludes that the fee allotted to schedule incentive would be better saved for cost reduction incentive, and in fact OSD procurement

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personnel state that the early emphasis on schedule incentives appears
to be diminishing. [14]

Several approaches have been used in quantifying the worth of
additional reliability and maintainability. Van Tijn developed a model
which measures variation in support costs as a function of subsystem
reliability and maintainability. [31] The model accumulates support
costs based on the maintenance effort predicted by reliability models of
the system. Douglas Missiles and Space Systems Division performed a
reliability sensitivity analysis prior to suggesting a reliability
incentive for a missile system. [32] Data derived from their study is
shown in Table 2.

Table 2

Missile Reliability	.84	.97	.99	.999
Operational Reliability	.67	.78	.80	.80
Cost/Missile (10^3)	230	275	311	355
Number Required	148	128	125	125
Total Missile Cost (10^6)	35.2	35.2	39.0	44.3
Total Operational Cost (10^6)	37.0	35.2	38.8	44.1

This allowed Douglas to determine the optimal reliability around which
the incentive would apply. A report by the Management Systems
Corporation contains an example in which incentives were desired for both
accuracy and reliability in a tactical ground-to-ground missile.
[16, p. 22] In this case the overall effectiveness criterion was the
ability to inflict a specified amount of damage on a given target with a
specified level of confidence. Thus the number of missiles procured
would depend on the number required to achieve the fixed level of damage,

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and accuracy and reliability could be directly related to the damage
expected per missile launched. Variations in accuracy and reliability
could then be evaluated by the cost savings in missile procurement, and
the incentive fee related to the cost saving incurred. When it is not
possible to relate the performance variation directly to a cost saving,
the Management Systems Corporation study advises:

> . . . it is necessary to estimate the relative dollar worth of
> effectiveness over the range of project outcomes. This judgement
> might be expressed as follows:

$$\Delta \text{ Value (\$)} \quad \boxed{\diagup}$$
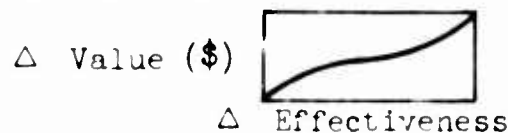
$$\Delta \text{ Effectiveness}$$

> The dollar value of effectiveness defines a tradeoff between total
> cost and effectiveness which the government wishes the contractor
> to use when making decisions which could affect both quantities.
> [16, p. 77]

Even comprehensive analysis prior to development contract negotiation
leaves many unanswerable questions. In the next section we will consider
how the ability to decompose a large project into smaller activities
can aid in obtaining an accurate cost-time-performance tradeoff surface,
but this capability does not aid appreciably in resolving the basic
uncertainties that exist in estimating system effectiveness. To decide
between all potentially important alternatives at the very initiation of
a program is at best a guided guess, and at worst the cause of an inade-
quate system or costly development redirection.

Cost, Time, Performance Tradeoffs

The Incentive Contract Guide confidently states:

> . . . one method for making this determination (of tradeoff functions)
> is to request the contractor to provide forecast performances for
> several combinations of cost and schedule, (with analysis by the
> government as to reasonableness). This of course will be standard
> practice under Project Definition Phase. [4, Ch. VIII, p. 7]

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The difficulty in obtaining such a performance forecast is that there are many equal cost methods of achieving very different final sets of specifications. A specified missile range may be attainable with alternative thrust capability, fuel capacity, and vehicle weight. Given only the cost and schedule the contractor usually will have wide latitude in the systems he can produce. However, the derivation of a realistic cost-time-performance surface is desirable in any development project. It is especially necessary to formulate a consistent incentive contract. And it may make possible submission of several FFP development bids with different specifications by the same contractor.

There are quantitative tools available and under development which yield estimates of the cost-time and cost-time-performance surfaces. A PERT network, for example, is established by defining a set of activities, an order or precedence relation, and a spread of activity completion times for some fixed funding. Methods have been available for some time to derive the expected completion time for the total project. The distribution of project completion time can be computed, analytically if the activity completion times are exponentially distributed, and by simulation in the case of more complex distributions. [30] If cost is a convex function of completion time for every activity, the minimum cost required to complete the project in a specified time may be derived.[9] [33]

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[9]Roderick W. Clark points out that time-cost data is not readily available because management accounting systems are not structured to provide ex ante estimates of costs as functions of completion times for various activities. This is due to the fact that accounting systems are designed to facilitate rapid reimbursement for incurred costs with only secondary emphasis on internal control. [34]

Thus, theoretically, the PERT network could be reformulated for several representative sets of performance specifications, and the general nature of the tradeoff surface approximated. However, even a system with a relatively small number of performance characteristics may have thousands of separate activities. Each time the key variables are assigned new values, the contractor must find the subsystems and sub-sub-systems that achieve the required performance levels with minimum cost for some fixed schedule. The requirement to reformulate in detail the entire PERT network would require an inordinate amount of administrative time, and would interfere with the normal work of the development personnel.

There are thus two separate problems. First the minimum cost subsystems that achieve desired performance in fixed time must be determined. Then the minimum cost schedule to develop specified subsystems must be found. Edward Zschau has formulated a procedure called "Project Modelling" which considers technical interrelationships among design specifications, precedence relationships among development activities, and determines the minimum cost of developing within an allotted time a system with specified performance characteristics. [17] The model assumes that cost in each activity can be expressed as a convex function of improving performance or decreasing development time, and that the total project cost can be expressed as the sum of the activity costs. Rather than attempting to solve the entire problem simultaneously, the procedure uses the tradeoff surfaces generated at activity levels as inputs to higher level optimization models and iteratively determines the minimum cost activity specifications for fixed time. For this set of activity specifications the minimum cost schedule is derived, and the specification

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model is then re-entered with the new schedule. Zschau has shown that

this iterative procedure converges to a solution optimal in the following

sense.

> When the project demands (e.g., the desired project duration and
> system performance characteristics) together with the activity cost
> functions are fed into the project model, it outputs the minimum
> project cost, the optimal engineering specifications, and the optimal
> project schedule for these demands and cost functions. By comparing
> the optimal project costs with the alternative sets of project
> demands that give rise to them, the optimal tradeoff functions
> relating time, cost, and performance can be generated. [17, p. 23]

Decentralized project management offers further advantages. Less

information flow is required, decisions are made closer to sources of

data, and it may be possible for the buying agency to decompose a large

development project into subprojects for several contractors. This would

allow each contractor to compete for each subsystem, and yet allow the

buyer to insure the proper meshing between subsystems. Furthermore the

contractor could furnish tradeoff surfaces without divulging either

activity costs or proprietary process information.

The feasibility of project modelling depends on the accuracy with

which development personnel will predict costs. Some incentive for

accurate estimation rather than protective estimation may be required.

The project modelling approach should be tested in a tractable develop-

ment program to determine its further applicability.

One additional consideration in besing decisions on cost-time-

performance surfaces is the uncertain nature of contractor control. For

example, top level program management may intend that a limited number

of development hours be devoted to reliability considerations, but

engineers at the working level may be motivated by professional or personal

interest to spend more thar the allotted number of hours on this area.

The time may be charged to an appropriate task, but the individual may still be thinking about ways to improve reliability. Overall project tradeoffs should be viewed as estimates to be systematically improved.

Profit

If system value and development cost were known as functions of availability date and performance characteristics, a consistent fee function could be defined by applying a profit rate to the difference between value and cost at any outcome. We have indicated that this would in general not motivate the contractor to aim at the outcome which maximizes monetary profit. However, the actual profit attainable at any outcome can be a strong determinant of contractor behavior if profit rapidly decreases as characteristics drop below those desired by the buyer. Our hypotheses about contractor behavior imply that contractors will be motivated away from areas of potential monetary loss, but are relatively indifferent between small positive profit increments. In August, 1963 the Weighted Guidelines revision to ASPR 3-808 was issued to aid contracting officers in determining appropriate profit rates. [11] Prior to this, contracting personnel were provided with a wealth of suggested factors and considerations, but were required ultimately to base their profit proposals on experience and judgement. In a report on actual contracting practices, Sumner Marcus states:

> Many contracting officers choose the expedient solution to their quandry (of conflicting and diverse decision elements). Through experience they arrive at a profit or fee rate that is well below the maximum permitted, but high enough so that the contractor will accept it, and they use these few rates over a long period for all contracts they negotiate, regardless of contractor situation. As time goes on they tend to lower the rate slightly to establish themselves as good bargainers. [35]

The Weighted Guidelines procedure establishes categories such as Contractor's Input to Total Performance, Contract Cost Risk, Past Performance,[10] and Special Profit Considerations. Contracting officers use available information to assign allowable weights to appropriate categories, and then rely on judgement in assigning weights to risk and special factors.

Based on interviews conducted prior to September, 1964, the most frequent criticism of Weighted Guidelines was that most government activities used the technique to establish a rationale for the final negotiated profit rate, rather than establishing an initial profit objective. Table 3 shows eleven FFP procurement actions negotiated at one buying activity using weighted guidelines.

Table 3

Weighted Guideline Objective	Profit Proposed	Historical Rate	Profit Negotiated
12.37	11.7	10.5	11.7
11.45	11.11	11.11	11.11
11.45	11.11	11.0	11.11
11.87	6.0	6.0	6.0
11.98	11.1	11.1	11.1
14.0	11.1	11.1	11.1
12.02	6.0	10.0	6.0
12.1	10.0	10.0	10.0
14.0	11.1	11.1	11.1
15.0	21.0	----	15.0
13.2	11.0	11.0	11.0

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[10]Collection and retention of contractor performance data is formally required by the Contractor Performance Evaluation program. Contractor success in meeting cost, performance, and schedule requirements will be recorded, and source selection boards will be required to review this information prior to making contract awards. [36]

This particular set of data confirms that contracting officers use high initial profit objectives to avoid later explanation of failure to negotiate at a point below or near a valid initial objective.

The contractor cannot press for too high a profit rate or target cost for fear of later unfavorable attention or the label of "profiteering." Or government procurement officer discussed the environment in which the contractor's negotiator must operate.

> It is quite evident that the lead negotiator's performance is staged in a "fishbowl" visible alike to corporate and division, but the end product of his negotiation is also subject to much scrutiny. In a government negotiation he can't afford to extract a profit too high even if the opportunity arises. Two of the interviewed companies recently had instances where they felt the profit was too high and the contract was not executed. In the only corporation interviewed that hired negotiators from outside the Government/ Aerospace complex, it was stated that the most difficult point to instill in their thinking was not to always seek the maximum profit as is customary in normal commercial bartering. [37]

When target costs are negotiated a good deal higher than projected cost outcomes, the contractor rarely risks taking a windfall profit, but instead voluntarily rebates profit, or commits additional funds for performance improvement, schedule acceleration, or related research. This behavior is explained not only by the existence of the GAO and Renegotiation Board, but by the low incremental cost of supporting additional research in this way.[11]

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[11]The Renegotiation Board annually reviews profits of contractors with more than $1 million in sales in order to recover excessive profits. The Board attempts to take the individual contract types into consideration, but it is difficult to determine those contracts in which profits are due to contractor efficiency as opposed to skill during negotiation, and the board's judgement has been questioned. [3, p. 261]

Recent DOD profit policy has aimed at negotiating higher profit rates in return for the contractor assuming more of the cost risk. The number of cost incentives negotiated in the 50/50 to 70/30 range has increased substantially, and there are currently very few low risk sharing arrangements concluded. [4, 6] This indicates the awareness at procurement policy levels that a number of factors motivate the contractor, and that by using higher profit at the desired outcome, and lower profit at less desirable outcomes, the government can motivate contractor behavior that is more consistent with government values.

Formulation of Contractual Incentives

A well structured incentive arrangement has two objectives. It will create a financial motive for the contractor to achieve superior performance in all variables. If this is impossible, the fee attainable at every possible outcome will reflect the government's ranking of that outcome, thereby guiding the contractor's choices.

The initial use of incentives was restricted to cost incentives, and the fees were almost always linear in the development cost. Justification for linear fees in such contracts is their simplicity of negotiation and ease of interpretation. The previously referred to Management Systems Corporation study justifies linear fees in terms of their flexibility. A contractor can always be motivated to aim at a certain outcome (if he is a fee maximizer) by means of a linear fee. But the example they present is misleading.

Let C_D = Development Cost,

C_o = Operating cost during lifetime,

C_T = Total Cost = $C_D + C_o$.

Assume that C_O is dependent on C_D in the following way:

$$pr[C_o = \xi \,|C_D] = C_D e^{-C_D \xi}$$

Then $E[C_o] = 1/C_D$, where E denotes the expectation operator. Hence $E[C_T] = C_D + 1/C_D$. Now assume that the buying agency desires to find that C_D which minimizes C_T. Then $\dfrac{\partial C_T}{\partial C_D} = 0$, which implies that $C_D = 1$.

Let f denote the contract fee as a function of development cost, and assume that the contractor maximizes expected fee. In order to motivate the contractor to set the development cost, C_D, at 1, we must have $\dfrac{\partial f}{\partial C_D} = 0$ for $C_D = 1$. If the fee is linear in the total cost, C_T, we have $f = \alpha - \beta C_T$, for some α, β, and $E[f] = \alpha - \beta E[C_T]$ or $E[f] = \alpha - \beta[C_D + 1/C_D]$. This is maximized at $C_D = 1$ for any choice of α, β. If however, the fee is linear in the development cost, C_D, (e.g., $f = \alpha - \beta C_D$), then the development cost chosen by the contractor is indeterminate without further assumptions.[12] Using linear fees, we must be careful about the variable the fee is based on, for the fee in the above example is certainly nonlinear in development cost.

When CPIF contracts were extended to include multiple incentives, the fees were generally independent, linear, and additive. For example, development cost might be weighted 40%, schedule 20%, and performance 40%. Individual performance elements such as thrust, or range, were then assigned portions of the 40% performance weight. This approach had

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[12] Recall that we were required to postulate the existence of a user cost function or disutility resulting from cost reduction to explain the contractor's behavior in this case.

several defects. It required the subjective ranking of the separate
performance, schedule, and cost elements. Any set of weights valid in
expressing government value at one point were not in general representa-
tive of value over the possible performance spectrum of the system. And
in such a fee arrangement it was possible for the contractor to earn
target fee while submitting poor performance in several variables. Con-
sider the following example which illustrates this fault of compart-
mentalized incentives.

Speed
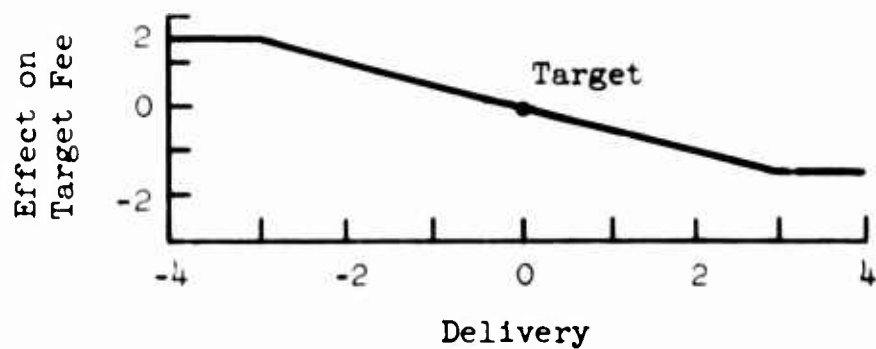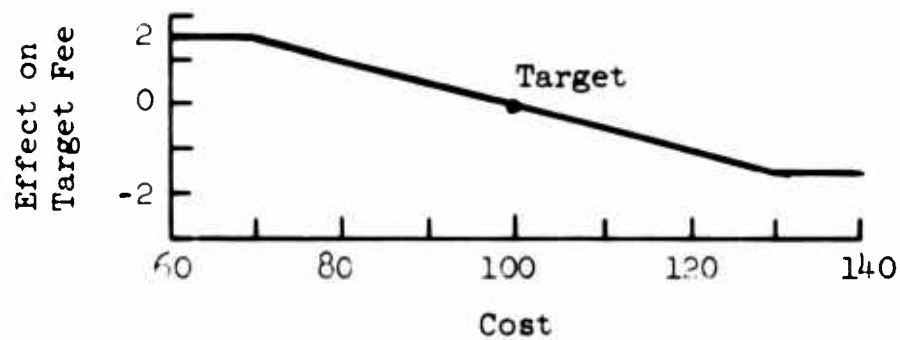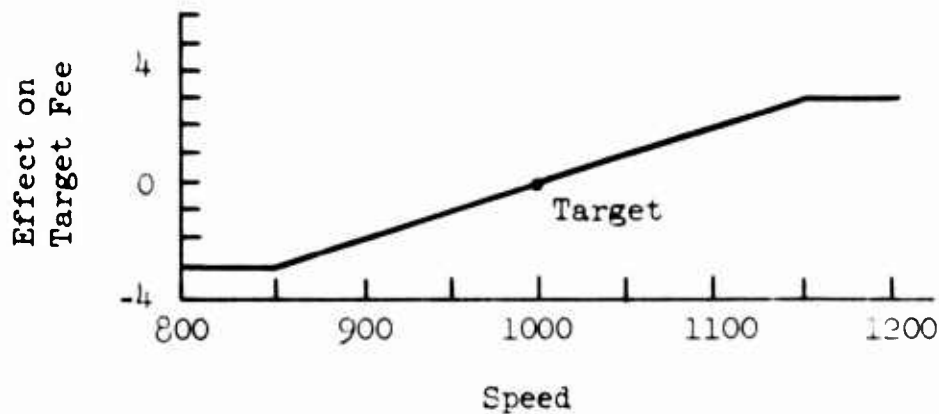
If the contractor delivered an 1150-mph aircraft three months after target at a cost of $130 million he would receive target fee, and if he delivered the same high performance aircraft six months late at a cost of $160 million he would still earn target fee. [2, p. 43]

The linearity of the fee structure oversimplifies the guidance to the contractor. When the performance level of some element is close to minimum acceptable, the value of an increment in that variable's performance is worth much more than an increment in performance in some variable already close to the maximum desired. Linear incentives do not reflect this. One partial solution to compartmentalization is to extend the range of the cost incentive to retain some control over cost overrun. This is termed "overlap." And the partial solution to the independence of the fees is to define the acceptable outcomes in terms of several variables. For example, acceptable minimum range might be 4000 miles, and acceptable minimum speed 1600-mph, but no more than minimum fee would be paid unless range plus speed exceeded 6000. This is simply taking care to bound the region within which a linear fee will be acceptable to the government.

Two methods have been proposed to define a fee surface which more
accurately reflects government values than do linear fees. The Logistic
Management Institute proposed a "Tabular Model" which uses a set of
multipliers to alter the fee structure in regions of contractor performance defined as "outstanding" or "poor." [15] After the government and
contractor negotiate the regions deserving special attention, fees are
structured independently for each variable, and total fee is the sum of
the individual fees. The fee for the outcomes in the specially designated
regions are multiplied by suitable constants, greater than one for
superior performance, and less than one for below standard performance.
This corresponds to modifying a planar fee surface by steeper planes in
these designated regions as shown in Figure 1.
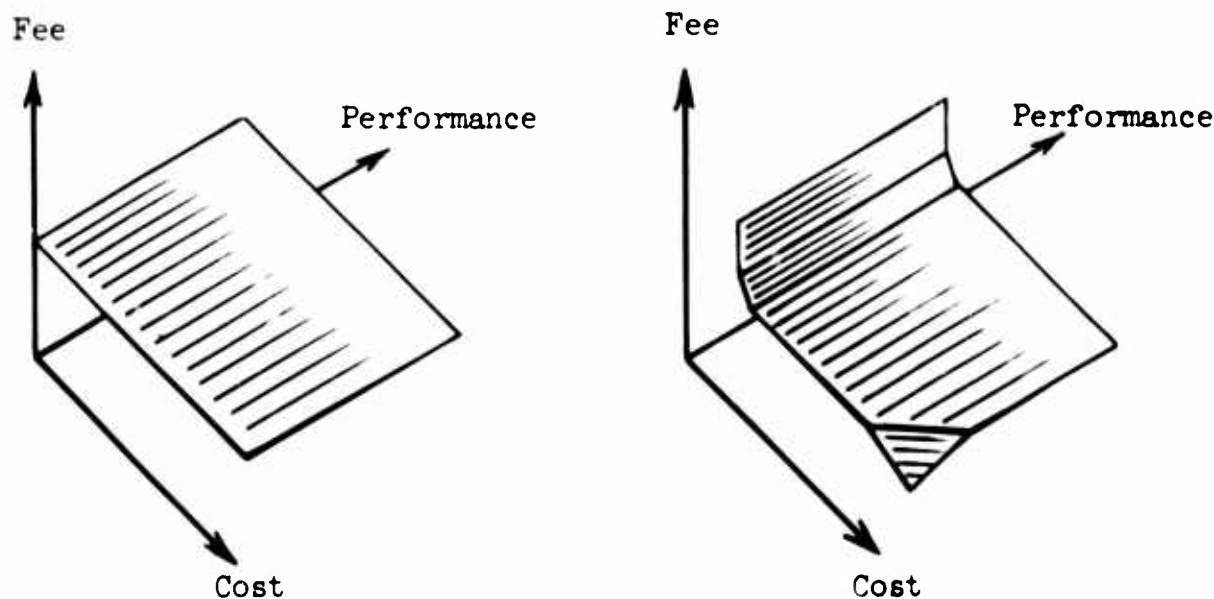
Figure 1

The name "Tabular" arises since the buying agency must define grades of
performance (tables) for the designated variables, and then must determine appropriate multipliers for each grade. For example, assume the
basic fee for control of development cost is linear, and the buyer desires

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to reward exceptionally low development cost and penalize a combination of high development cost and low performance. The buyer therefore might divide the range of expected development cost outcomes into ten grades. The lowest two of these grades could be assigned the multipliers 1.4, and 1.2, while the combination of high development cost and low performance could be assigned the multiplier 0.5. The use of multipliers is an explicit method of requiring the procuring agency to determine its own utility function, and then introduce any significant nonlinearities into the incentive fee. Using multipliers for combinations of variables requires a method of ranking the relative desirability of outcomes. Without a model of system effectiveness, determining the relative value of the outcome of three parameters, each having ten grades, could be tedious.

We have seen that a basic problem in incentive fee formulation is that the effectiveness of a system is not additive, but is a complex function of the variables characterizing the system. The Management Systems Corporation study recognizes this and suggests that effectiveness be measured by a cost-effectiveness model, and that the incentive fee be a monotonically increasing function of government value. The fee structure they propose is:

$$f = f_t + \alpha(\Delta V - \Delta C) ,$$

where   f = final fee,

   f_t = target or initial fee,

   α = sharing proportion,

ΔV = incremental increase in effectiveness value over target effectiveness value,

ΔC = incremental increase in cost over target cost.

We discussed the problem of translating system effectiveness into dollar terms. Even when it is difficult to make such comparisons they will be implicitly used in any case, and the use of cost-effectiveness techniques requires that they be made formally. One of the criticisms heard against government contract management is that the buying agency is vague about its requirements and choices among performance capabilities. This objection could be met by furnishing the contractor with the cost-effectiveness model, or data derived from it. The contractor would then have a more objective basis for tradeoff decisions, and the government would have a more objective method of deciding whether the contractor did, in fact, act in the government's best interest.

A combination of Zschau's approach to decentralized project management, the periodic updating and use of system effectiveness models, and use of an award fee appear to offer substantial efficiencies in development contracting. An initial cost-effectiveness model may allow an incentive structure that reflects government desires. But as development progresses and both the cost-time-performance surface and the external environment change, the most desirable system will also change. By using project models and system models capable of being updated, the contractor and buying agency could respond to the operating environment adaptively. If tradeoffs are to be evaluated in detail, decentralization would be as desirable in the user's cost-effectiveness model as in the development project model. Decentralization allows more detailed and accurate

estimates of effectiveness value but requires a higher level model to avoid suboptimization. Use of an award fee is a step closer to the "After-The-Fact-Evaluation" contracting method proposed by Scherer.[13] The award fee allows the buying agency to take account of outstanding innovation or management, and to compensate for unforeseen requirement changes or technical problems that may have precluded the contractor from earning a fair profit in an otherwise satisfactory program.

Implementation of such changes would have to be gradual. The defense and aerospace contractors are just becoming familiar with the multiple incentive contracts initiated in 1962. After an initially lukewarm reception, industry reaction now appears favorable. [13] A Douglas Missiles and Space System Division report states:

> An incentive program usually stimulates the development of efficient techniques resulting in higher reliability and more contract dollars. [32]

Contracting officials at two aerospace corporations commented that the magnitude of the incentive fees did not seem to be very important. The knowledge that the project was "on incentive" was sufficient to induce higher performance and better planning. Some corporate executives felt that performance incentives were superfluous. Other writers argue that reliability and maintainability incentives are required with cost incentives to prevent corner cutting in quality. W. C. Frederick states:

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[13]Under Scherer's proposal the government would maintain data describing contractor efficiency in cost control, reliability, operational performance, and adherence to schedule, which would be used in determining profit rates, and awarding contracts. In addition, Scherer proposes that the growth, long term profits, and direction of activity of the contractor be planned by a Performance Evaluation Board. This would place emphasis on the long range incentives of survival and growth. [3, p. 327]

The risk to the purchaser (in a firm-fixed-price contract) is a bit more subtle, but no less real. Under the FFP type of contract the contractor's profit varies inversely with the cost of the system; therefore he has a tremendous incentive to cut costs, perhaps to the point where the quality of the system is degraded. Thus unless the specification or work statement is extraordinarily precise and complete, the purchaser may be forced to accept a product of a quality lower than he had expected. [39, p. 7]

And again in arguing against single dimensional (cost) incentive contracts,

The provisions of one incentive contract of which the author is aware were such that the contractor's fee would be greatest if his cost were exactly $1,000,000. It was the stated intent of the program manager to accumulate costs of $1,000,000, and since costs were in danger of exceeding that amount, the planned reliability program was curtailed accordingly--despite the fact that the reliability objective was conceded to be extremely important to the customer and difficult to meet. [39, p. 8]

Scherer, however, repeatedly claims that weapons system contractors have an extremely high aversion to cutting quality or reliability to gain short run profit. The incentives toward high quality--corporate reputation, the reluctance of the purchasing agency to authorize design changes that impair quality, and the possible effect of low quality on future awards--do not appear applicable to commercial firms, while Frederick's comments, directed at defense contractors, do appear to be more applicable to non-defense firms with several markets. The utility of current profit is higher for these firms, and the renegotiation constraint is not present.

Government procurement personnel are pleased with the results of contractual incentives. The Incentive Contract Guide states:

The success of the program (expanded use of performance incentives) seems clearly revealed in performance levels achieved at target or high than target in the majority of cases and cost being controlled in almost every instance within 10% of negotiated target costs. [4, Ch. VIII, p. 2]

It is not clear whether these result. have been obtained because of the existence of incentive fees, or because of the detailed system analysis and capability estimation that precedes contract negotiation. A set of experiments in which contractors submit proposals for FFP, CPIF, and CPAF awards would illuminate the preferences for risk and contract type within individual firms, and between firms.

Summary

We have indicated the important trends in contracting for the development and procurement of complex weapons and space systems. The first is the greater selectivity and limited production instituted by Secretary of Defense McNamara, which creates a competitive atmosphere among contractors, and places the government in a more powerful bargaining position. Thus the government is able to institute management controls and practices which it considers conducive to the more efficient use of procurement resources and development personnel. The second operative factor is the growing bank of data recording contractor performance. Over a period of time chance performance variations will be averaged out, and the long term management competence which a given contractor brings to his projects will be quantitatively demonstrable in terms of development costs versus estimates, and field performance versus projected performance. The final factor is the growing arsenal of management techniques and tools for control and analysis of alternatives in project definition and development. Adaptive project modelling by the contractor and a similar government effort could operate together to allow sequential decision making during the development process. This

would be more efficient than fixing the ranges of system performance far in advance of any firm information regarding desirability, feasibility, and cost.

These three factors interact to allow government initiation of the project modelling technique, by creating general system models and project models of various space and weapons systems. Contractor performance data would provide initial model data which could be continually updated by completed projects. Similarly the system models could be refined by operational testing. As government and industry gain familiarity with the formulation, use, and properties of system models, the government could ask for development program bids which would have tradeoff decisions made by the government, sequentially, with the profit award being negotiated after project completion with final cost, performance, risks, and special factors available for consideration. We feel that this approach to program management promises a more responsive contractor effort in research and development than do the more rigid contractual incentives.

CHAPTER II

UNCONSTRAINED CONTINGENT PRICING MODELS

Introduction

In reviewing the use of incentives in development we observed that
performance incentives were intended to motivate superior technological
progress in the design and development of new products. Once high per-
formance features are designed into equipment, production inefficiencies
can degrade operation, but even outstanding production practices cannot
improve uninspired design. We also noted that the difficult problems
in designing contractual incentives were encountered in the advanced
system area. It is useful to point out that relatively few contractors
are involved in large advanced system development. The majority of pro-
ducers supply standard or slightly modified equipment, and generally do
not sell on an incentive basis. Government practice is to procure
noncomplex products (shoes, medical supplies, hand tools, ammunition) in
large lots and to specify the Acceptable Quality Level (AQL), or percent
defective allowable in the lot.[1] The acceptability of complex items such
as communication equipment or vehicles is determined by defects-per-unit,

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[1]Sampling of incoming lots is conducted in accordance with approved
sampling procedures. A fixed price is paid for accepted lots. Items
which cannot be designated defective or nondefective (such as capacitors
with a range of capacitance, or ammunition propellants with variable
muzzle energy) are classified by conformance to specification, and lot
disposition is determined by the percent of the lot estimated to fall
outside tolerance limits. [41]

demerits-per-unit, or reliability tests such as the AGREE procedures.
[22]

In production, as opposed to development, there is less cost un-
certainty and little cause for changes in product specifications. Con-
formance to these specifications can be more accurately determined than
in development, and it appears that fixed-price-quality-incentives, or
"contingent pricing contracts" are applicable to these production situ-
ations. A contingent pricing contract would provide for variable payment
based on the degree of product conformance to design specification as
measured by an agreed on testing or sampling method. Such adherence to
specifications is commonly termed "quality" and contingent pricing is
therefore more of a quality related concept than are performance incen-
tives in general. Note that in discussing performance incentives we
implicitly assumed that at the completion of product development the
performance parameters were precisely known. In fact these parameters
(MTBF, range) can only be estimated or predicted from the observation of
other variables. Hence there is sampling uncertainty in the payment
received by the contractor or producer. Contingent pricing policies are
designed to handle this sampling uncertainty directly. The concepts
involved in analyzing the quality situations are also applicable to
structuring performance incentives based on measured performance
variables.

When the buying agency samples an incoming lot or tests an incoming
equipment and finds that it does not conform to specification, it may
either waive the nonconformance or reject the lot or item. When items
cannot be rejected due to delivery or urgency considerations a single

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unit price is not an effective tool to motivate superior production performance. The item marginally acceptable may be improved later by supplier effort, but the extra time and possible government expense involved should detract from the unit price in some way. Contingent pricing provides a method of rewarding contractor excellence in production and penalizing poor contractor quality control.

There have only been three models that explain contractor response to performance incentives, and of these only two attempt to design incentive policies optimal in some sense. In this chapter we briefly discuss these models, state the assumptions in them which we feel prevent the derivation of realistic policies, and then in the next chapter develop a method for generating more realistic incentive pricing policies.

Hill Model

The forerunner of the contingent pricing models is the effort by Hill [42] in 1960 to explain the incentive effect of both payment and acceptance sampling plans. Hill suggested that the primary purpose of acceptance sampling was to motivate higher production quality rather than simply detect and reject deficient batches. He assumed as axioms:

(i) the distribution of the batch percent defective submitted to the purchaser is entirely dependent upon the actions of the manufacturer; and

(ii) the manufacturer will take those actions which maximize expected profit.

From these axioms we see that if the lot quality (percent defective) submitted is to improve, the manufacturer must take action, and that he

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will do so only if it is economically advantageous to him. A further implication is that it should be to the manufacturer's economic advantage to offer good quality and to his disadvantage to offer poor quality. The sampling and payment schedules to which Hill restricts his discussion provide for a fixed payment if the lot is accepted and no payment otherwise. In situations of this type, higher quality should be rewarded by greater frequency of lot acceptance. While all operating characteristics produce this result, it is standard practice to shift sampling plans as the sampling results provide information on the average percent defective submitted in incoming lots. Consider the following case which illustrates how certain features of the Dodge-Romig sampling tables operate against good quality. [43]

Let   LTPD = 4.0%

        Consumer's Risk = 0.10

            Batch Size = 500.

OC curves for four different plans are shown in Figure 1, and each curve is to be used for a particular range of process average. In Hill's example producer A submits lots with an average 0.4% defective, B submits 0.5% defective, and C submits 1.3% defective. Yet A will have 80.2% of his lots accepted, B will have 92.5% accepted, and C 86.7% accepted. The lack of incentive to good production is obvious.
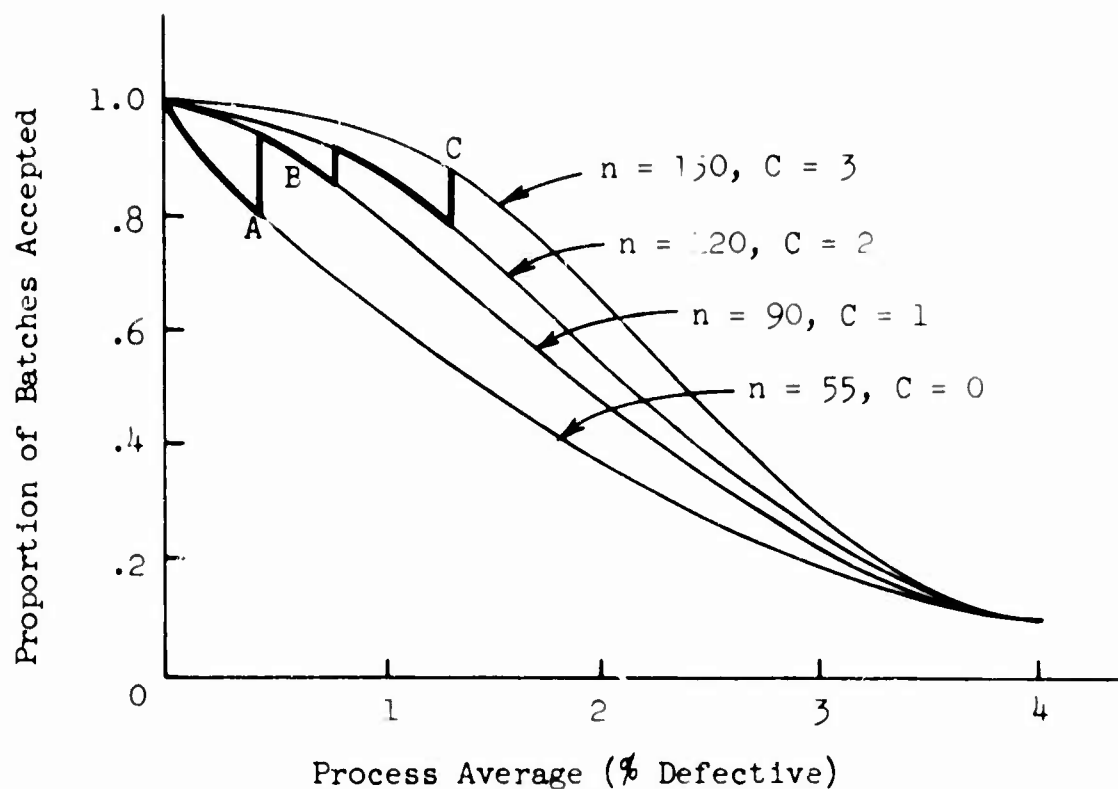
Figure 1

Hill outlined a basic economic theory of sampling inspection to account for the incentive features. He assumed

1. the distribution of defectives in submitted lots is binomial with parameter p, and

2. the unit cost of manufacture as a function of outgoing quality, h(p), can be represented by $K/\sqrt{p}$.

Let   x = number of defectives observed in the sample,

   N = batch or lot size,

   LN = price paid per accepted batch,

   T = average profit to the producer per batch,

   n = sample size,

   a = acceptance number,

   P = proportion of accepted batches.

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Since x is binomial, $P = \sum_{x=0}^{a} \binom{n}{x} p^x(1-p)^{n-x}$. Under the assumption that a rejected batch is a total loss to the manufacturer, $T = NLP - Nh(p)$, or

$$T = KN\left[\frac{L}{K} P - \frac{1}{\sqrt{p}}\right] .$$

Given K, L, and a, the manufacturer aims to adjust p to maximize T. We can derive the relationsnip the optimal p must satisfy. First

$$\frac{dP}{dp} = \frac{d}{dp} \sum_{x=0}^{a} \binom{n}{x} p^x(1-p)^{n-x} = \frac{-n!}{a!(n-a-1)!} p^a(1-p)^{n-a-1}. \text{ Then}$$

$$\frac{dT}{dp} = KN\left[\frac{L}{K} \frac{dP}{dp} + \frac{1}{2\sqrt{p^3}}\right] = 0.[2] \text{ Or } \frac{L}{K}\left[\frac{n!}{a!(n-a-1)!} p^a(1-p)^{n-a-1}\right] = \frac{1}{2\sqrt{p^3}} .$$

Figure 2 indicates the variations in producer profit, T, with submitted quality under a sampling plan in which $n = 150$, and $a = 11$. The plan is a MIL-STD-105A plan for an AQL of 4%.

Hill notes that the variation of the optimal p is relatively insensitive to changes in the price-cost ratio. Furthermore the sampling plan used causes the producer's optimal choice of p to be in the neighborhood of 4%, the quality the plan is designed for. Hill did not extend his remarks beyond this example, but his essential points are clear for the purchasing agency. If the producer attempts to maximize expected profit through choice of quality level, both the sampling plan used and the price paid per lot will determine this choice.

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[2]To insure that T is actually maximized the producer would in fact have to test the end points and all interior points at which $dT/dp = 0$.
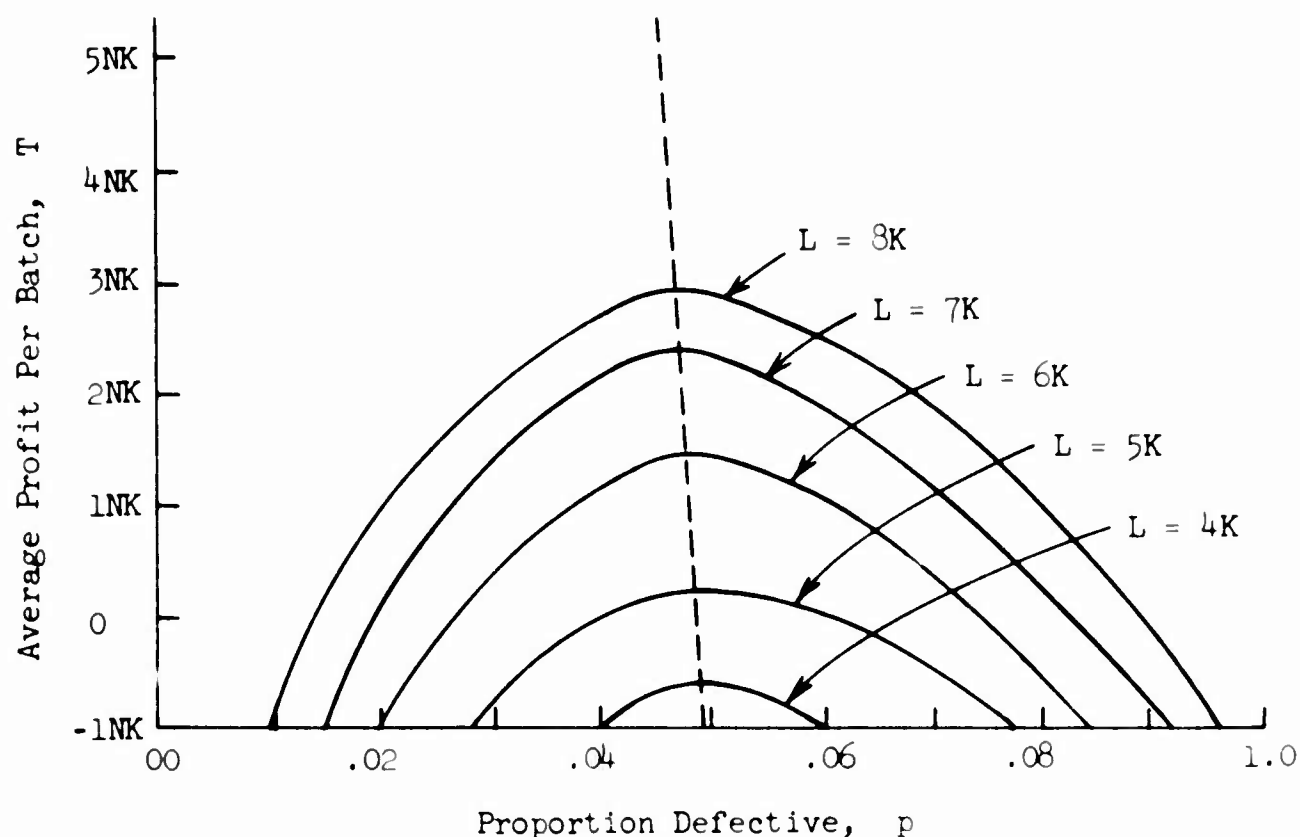
Figure 2

Johns-Lieberman Model

In 1961 Vernon M. Johns and Gerald J. Lieberman of Stanford
University formulated the contingent pricing problem in a more general
setting. [44] They described a situation in which the buyer agrees to
purchase batches of items, for a price to be determined in a prescribed
way according to the results of a sample of items from the lot. The
basic assumptions in the Johns-Lieberman model were that

1. the producer is capable of controlling the quality of his
    product to a known extent and at a cost known to both producer
    and purchaser, and

2. given any sampling plan and pricing policy the producer will
    choose the quality to maximize expected profit.

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They further assumed that the cost of sampling is charged to the buyer

and that the buyer chooses the sample size in advance. Johns and

Lieberman note,

> The situation is then formally that of a two person non-zero-sum
> game in which the strategy of one player (i.e., the pricing policy
> of the consumer) is revealed to the other player (the producer) in
> advance of his choice of a strategy. This type of game has a well
> defined notion of a solution. The consumer knows the producer will
> adopt a strategy (quality level) which will net him (the producer)
> the maximum expected return under the pricing policy chosen by the
> consumer. The consumer must therefore select the pricing policy
> which will net him the greatest return under the producer's
> corresponding optimal strategy.

The following example illustrates the model and certain problems that

arise in this formulation.

Let N = number of items in a batch,

p = probability each item produced is defective,

$h(p)$ = unit cost of producing items at an average quality level p, where $h(p)$ includes a normal loading for unit profit,

n = sample size,

x = number of defectives observed in the sample,

c = sampling cost per item,

$N^{\varphi}(x)$ = price paid for the lot when x defectives are observed,

T_c = expected consumer profit,

T_p = expected producer profit,

$g(p)$ = expected value of $\varphi(x)$, given n and p. (If x is a binomial random variable

$$g(p) = \sum_{o}^{n} \varphi(x) \binom{n}{x} p^{x}(1-p)^{n-x}.)$$

For simplicity in exposition let the value to the consumer be one per non-defective item, and zero per defective item. Denote the expected unit value by $V(p)$. In this case $V(p) = (1-p)$. The expected net returns to consumer and producer are then

$$(2.1) \qquad T_c = N(1-p) - Ng(p) - nc , \quad \text{and}$$

$$(2.2) \qquad T_p = Ng(p) - Nh(p) .$$

Since a unit profit is included in the $h(p)$ function, we assume that the producer will agree to a contract if $T_p \geq 0$ for some p. For fixed n T_c will be maximized when $Ng(p)$ is minimized, and from this we see that in an optimal policy we would like

$$(2.3) \qquad Ng(p) = Nh(p) \quad \text{for some p.}$$

At that p,

$$(2.4) \qquad T_c = N(1-p) - Nh(p) - nc .$$

For any n, (2.4) will be maximized by the p^* for which $(1-p) - h(p)$ is a maximum. In this case p^* satisfies $h'(p^*) = -1$. The situation is shown in figure 3. Consumer profit, T_c, will be maximized by using the smallest sample size, n, at which the producer can be motivated to produce at p^*.
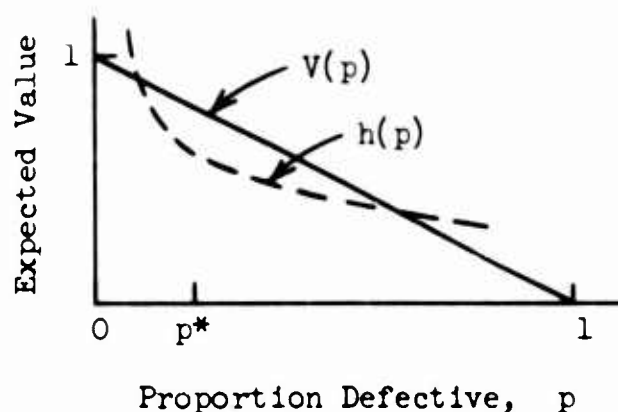
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Figure 3

Consider the case $n = 1$. To motivate production at p^* T_p must attain its maximum at p^*, or

$$(2.5) \qquad \frac{d}{dp} [Ng(p) - Nh(p)] = 0 , \quad p = p^* .$$

If x is a binomial random variable, we have from the definition of $g(p)$,

$$(2.6) \qquad g(p) = \varphi(0) \cdot (1-p) + \varphi(1) \cdot p ,$$

and from (2.5) $-\varphi(0) + \varphi(1) = h'(p^*)$, while from (2.3) $\varphi(0)(1-p^*) + \varphi(1) p^* = h(p^*)$. Solving for $\varphi(0)$ and $\varphi(1)$ we obtain the pricing schedule,

$$(2.7) \qquad \varphi(0) = h(p^*) + p^* , \quad \text{and}$$

$$(2.8) \qquad \varphi(1) = h(p^*) + p^* - 1 .$$

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A sample of one produces an optimal policy! Consider the following numerical example based on the preceding development and illustrated in Figure 4. Let tne value to the customer, $V(p)$, be $(1-p)$ as before.

Let $h(p) = \dfrac{.022}{\sqrt{p}} + .4$, unit production cost, and

$\quad\quad c = .05$, sampling cost,

$\quad\quad N = 100$, lot size.

We determine optimal quality, $p^* \approx .049$, and $h(p^*) = 0.495$. Therefore
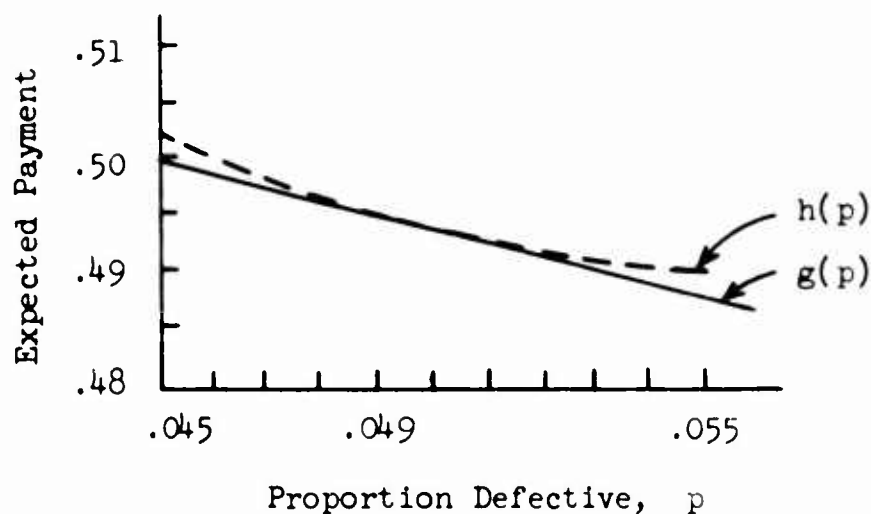
$$\varphi(0) = 0.544 \, , \quad \text{and}$$

$$\varphi(1) = -.456$$

Figure 4

This policy yields an expected consumer gain, T_c, of 49.55 and no producer gain beyond the unit profit included in the $h(p)$ function. That is, $g(p^*) = h(p^*)$.

Note that if the sample of size one is defective the producer incurs an actual net loss of $(.495) \times 100 + (.456) \times 100 = 95.1$, while if the sample is non-defective his actual gain is only 4.9. This is based on an investment of close to 49.5, and while tne expected profit and actual

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profit are acceptab'e at p*, we feel that it is unlikely that a producer would agree to this type of policy.

One of the elements we feel is intuitively unsatisfying is the magnitude of the penalty levied against the producer when a defective is discovered. We will discuss the implications of bounding the minimum payment in a later section, but to illustrate that lower bounds do not provide the solution, consider the producer to have no fixed costs, so that his unit production cost, $h(p)$, is $\dfrac{.022}{\sqrt{p}}$, and restrict the payments, $\varphi(x)$, to be non-negative. The optimal p^* is still 0.049, but $h(.049) = .1$. Let a sample of nine items be drawn and the price schedule be as follows:

$$\varphi(x) = 0.175 \ , \quad x = 0$$
$$\varphi(x) = 0.0 \quad , \quad x > 0 \ .$$

The producer is still motivated to produce at p^*, (T_p is maximized at p^*), and expected producer profit at p^* is 1.10. One defective observation in this case is not as disastrous for the producer as in the previous case since he loses only his investment and is not penalized additionally. However Figure 5 indicates that $g(p)$ is very close to $h(p)$ in the vicinity of p^*, and therefore the producer is not severely penalized for quality poorer than p^*. For example, if the producer chooses to produce at $p = 0.10$ rather than $p = 0.049$ his expected net loss is only .175. He will obtain an actual gain of 10.5 39% of the time, and will lose 7.0 61% of the time.
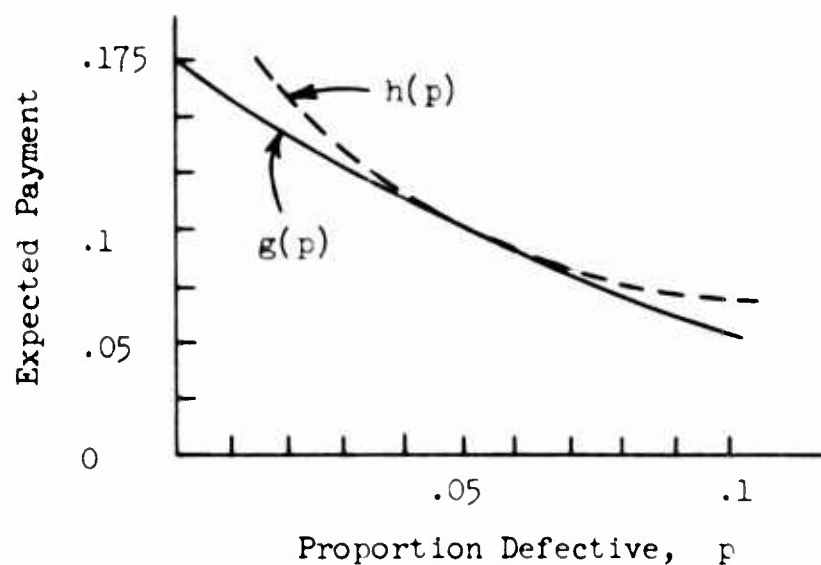
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Figure 5

A producer prone to corner cutting or gambling might find this policy attractive, and we suspect that a buyer would want tighter discriminatory ability.

Flehinger-Miller Model

The last numerical example used a sample size of nine and a single payment of 0.175 if no defectives were observed. This was not arbitrary, but was an optimal policy in a way that will now be described. In 1964 Betty J. Flehinger and James Miller reported a "product improvement" model very similar to the Johns-Lieberman model. [45]

They assume that a producer is already manufacturing at some verified quality level and is presumably receiving an acceptable fee. The producer can spend additional funds to improve the product quality. If this improvement benefits the consumer more than it costs the producer, such improvement will be advantageous to both parties. An acceptance test

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will be performed to verify that the product improvement has been achieved.[3] The fact that product improvement is advantageous to both parties prompts Flehinger and Miller to seek only admissible policies— those that maximize joint profit.

The important assumptions in this formulation not included in the previous models are:

1. If the producer makes no attempt to improve his quality it will be characterized by the basic quality at which he is currently producing, p_0.

2. A strategy or policy, $(n, \varphi_0, \varphi_1, \ldots, \varphi_n)$[4], or (n, φ) is defined to be __admissible__ if it yields a positive expected profit to both consumer and producer, and if no other policy yields a greater expected profit to both sides.

Assume the expected value to the consumer is $V(p) = 1 - p/p_0$. The expected profits can then be written as in (2.1), (2.2) as functions of p, φ and the sample size, n:

$$(2.9) \qquad T_c(p,\varphi,n) = NV(p) - Ng(p) - nc , \quad \text{and}$$

---

[3]Flehinger and Miller feel that this test is motivational rather than informative, since the test is the mechanism by which the producer is motivated to a particular quality. Actually the test is informative to the consumer. While the producer is assumed to have perfect control and can set the quality precisely where he chooses to, the consumer has no protection other than the assumed rationality (i.e., profit maximization behavior) of the producer. Hence the acceptance test both motivates the producer, and gives the consumer some indication of the quality actually attained.

[4]We will denote the unit prices both by $\varphi(x)$ and φ_x.

$$(2.10) \qquad\qquad T_p(p,\varphi,n) = Ng(p) - Nh(p) \ .$$

Given any policy (n,φ) the producer will choose p to maximize (2.10). Denote this p by p^*. We may express p^* as $p^*(\varphi,n)$, and express the consumer and producer profits at p^* as

$$T_c^* = T_c(p^*,\varphi,n) \ , \quad \text{and}$$

$$T_p^* = T_p(p^*,\varphi,n) \ .$$

Each (n,φ) is thus mapped into a point (T_c^*,T_p^*) and by Assumption 2 a policy is admissible if it maps into a point (T_c^*,T_p^*) such that $T_c^* \geq 0$, $T_p^* \geq 0$ and no other policy maps into a point (T_c',T_p') that dominates (T_c^*,T_p^*). This implies that an admissible strategy maps into a point (T_c^*,T_p^*) such that $T_c^* + T_p^*$ is maximum over all policies leading to the same value of T_p^*.

Since $T_c^* + T_p^* = N[V(p) - h(p)] - nc$, we see that by seeking admissible strategies, we arrive formally at the relation (2.4) which says simply that if a quantity is to be divided with a fixed amount going to one party, the other party maximizes gain by finding the point at which the initial quantity is greatest.

Flehinger and Miller derive the following procedure to find the set of all admissible strategies:

a. Find the p^* which maximizes $V(p) - h(p)$,

b. Out of the class of all policies, (n,φ), select those for which T_p is maximum at p^*.

c. Of these policies select those for which $T_c^* \geq 0$, and $T_p^* \geq 0$.

d. Classify the policies selected by the value of T^*_p, and out of all policies yielding the same value of T^*_p, choose those for which the sampling cost is minimum. These remaining policies are admissible.

We present an example to illustrate the method.

Let p = the probability that each item produced is defective,

$$(p \leq p_o)$$

N = lot size,

n = sample size,

$N^{\varphi}(x)$ = price paid for the batch,

$NV(p) = N[1 - p/p_o]$, expected batch value,

$h(p) = h_1 \ln(p_o/p)$ the unit cost of improving quality to p,

c = unit cost of sampling,

$q = 1 - p$.

To obtain the p that maximizes $V(p) - h(p)$ in this example we set $\frac{d}{dp} [V(p) - h(p)] = 0$ or $-\frac{1}{p_o} = \frac{-h_1}{p^*}$ and hence,

$$(2.11) \qquad\qquad p^* = p_o h_1 .$$

To find the policies (n,φ) which maximize T_p at p^*, set $\frac{d}{dp} [Ng(p) - Nh(p)] = 0$ for $p = p^*$, which is

$$\sum_{x=0}^{m} \varphi_x \frac{d}{dp} \binom{n}{x} p^*(1-p)^{n-x} + \frac{h_1}{p} = 0 \quad \text{for} \quad p = p^*, \text{ or}$$

$$(2.12) \qquad \sum_{x=0}^{m} \varphi_x \binom{n}{x} p^{x-1}q^{n-x-1}[x - np] = \frac{-h_1}{p} \quad \text{for} \quad p = p^* .$$

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This is a condition that $\varphi(x)$ must satisfy for every n in order that T_p will be maximum at $p* \epsilon (0,1)$.[5]

Consider all policies of the form $\varphi(0) = L$, a constant to be determined, and $\varphi(x) = 0$, $x > 0$. (2.12) then reduces to $\varphi(0) = \dfrac{h_1}{p*nq*^{n-1}}$, or since $p* = p_0 h_1$,

$$(2.13) \qquad\qquad L = \frac{1}{p_0 nq*^{n-1}} \quad ,$$

and $g(p)$ is then given by $\dfrac{q*}{np_0}$. We thus obtain

$$(2.14) \qquad\qquad T_c^* = NV(p*) - Nq*/np_0 - nc \quad, \quad \text{and}$$

$$(2.15) \qquad\qquad T_p^* = Nq*/np_0 - Nh(p*) \quad.$$

For this special class of policies we now choose sample sizes, n, that yield $T_c^* \geq 0$, and $T_p^* \geq 0$. Since L is a function of n, we first determine the minimum n such that $T_c^* \geq 0$. Denote this by n_1. This is given by

$$(2.16) \qquad\qquad n_1 \geq \frac{NV(p)}{2c} - \frac{1}{2}\sqrt{\left|\frac{NV(p*)}{c}\right|^2 - \frac{4Nq*}{p_0 c}} \quad.$$

---

[5] We remark again that every policy, (n,φ), must be tested at the endpoints, $p = 0$, $p = 1$, and at every interior point for which the derivative $= 0$. If any policy yields more than one point at which the derivative $= 0$, an additional constraint can be used to find a policy which does maximize T_p at $p*$.

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Denoting by n_2' the largest n such that $T_p^* \geq 0$, we obtain

$$(2.17) \qquad\qquad n_2' \leq q^*/p_o h(p^*) .$$

The requirement that for a fixed value of T_p^* we choose those policies (n, L) such that $n \cdot c$ is minimum implies that the forward difference of T_c^* with respect to n must be non-negative. Therefore n_2 may not be given by (2.17) but rather by $\Delta n[NV(p^*) - Nq^*/np_o - nc] \geq 0$, or

$$(2.18) \qquad\qquad n_2''(n_2'' + 1) \leq Nq^*/cp_o .$$

Hence n_2 is given by $\min(n_2', n_2'')$. This is shown in Figure 6.
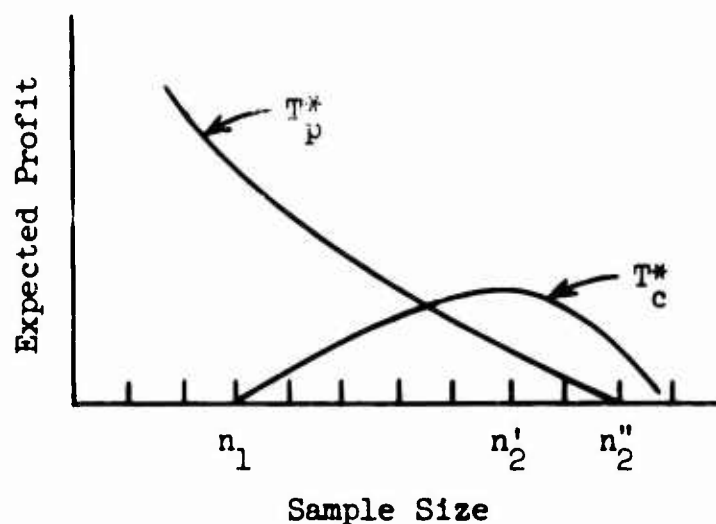
Figure 6

To summarize, all admissible policies of the special form $\varphi(0) = L$, $\varphi(x) = 0$ $(x > 0)$ are given by

$$(2.13) \qquad\qquad L = 1/p_o nq^{*^{n-1}}$$

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for n in an interval (n_1, n_2) where n_1 is given by (2.16), and n_2 satisfies both (2.17) and (2.18). This special class of policies will be proven admissible, and except for one special case exhausts the class of admissible policies!

The special case arises in the following way. The assumption 1 and the cost function, $h(p)$, imply that even if no improvement effort is made by the producer, quality will remain at level p_o, and at that level there is a positive probability, $(1-p_o)^n$, that zero defectives will be observed in a sample of size n. We have observed that $T_p(p^*, L, n)$ decreases in n, and eventually becomes negative. However, $T_p(p_o, L, n)$ never becomes negative, which indicates that in some cases there may exist a sample size, say $\tilde{n}$, at which the producer will choose to forego product improvement. This situation is illustrated in Figure 7 and Figure 8.
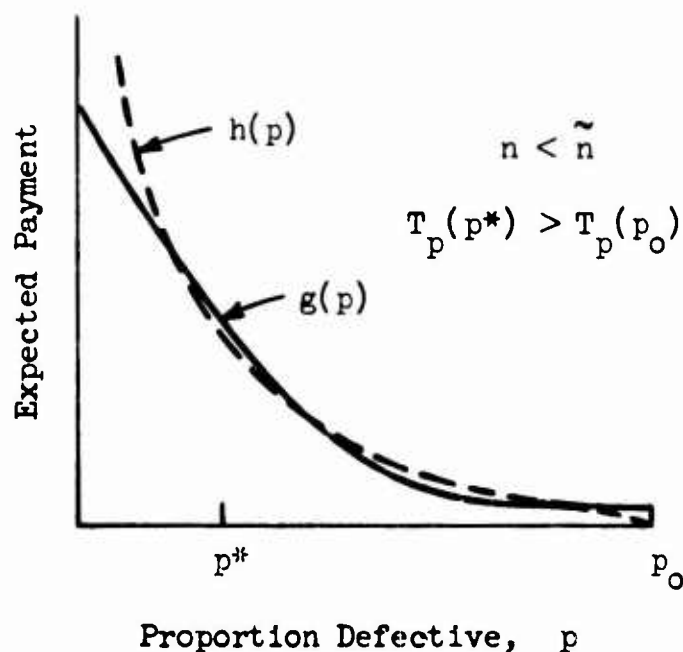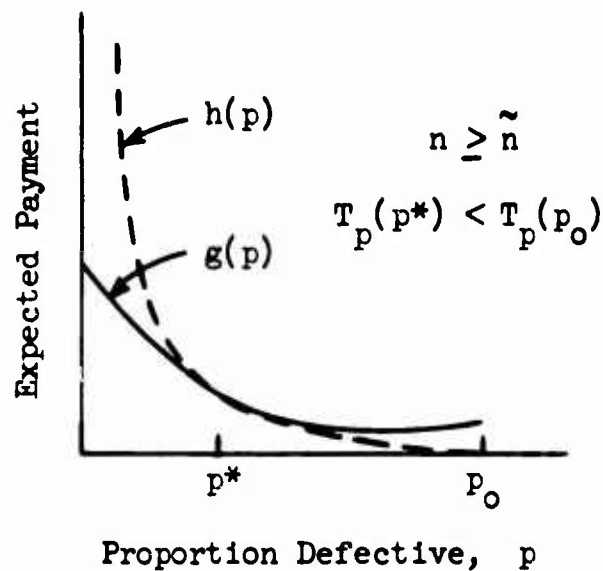

Figure 7

Figure 8

Flehinger and Miller constrain this situation from occurring by having the upper bound on sample sizes for the policies (n,L) also satisfy the condition that $T_p(p^*,\varphi,n) \geq T_p(p_0,\varphi,n)$, and if necessary, add an extra payment $\varphi(1)$ at sample sizes above $\tilde{n}$, that "remotivate" the producer to p^*.

The following numerical example illustrates the results just derived.

Let $\quad N = 100$

$$p_0 = 0.1$$

$$h(p) = .1 \ln .1/p$$

$$c = 0.4$$

From (2.11) the optimal quality, p^*, is 0.01, and the unit production cost, $h(0.01) = 0.23$. From (2.13) $L = 10/n(.99)^{n-1}$. (2.16) yields the minimum sample size, n_1, of 11. (2.17) yields $n_2 = 43$, and (2.18) yields $n_2 = 49$. Several of the admissible policies are listed in Table 1, and the situation is displayed in Figure 9.

Table 1

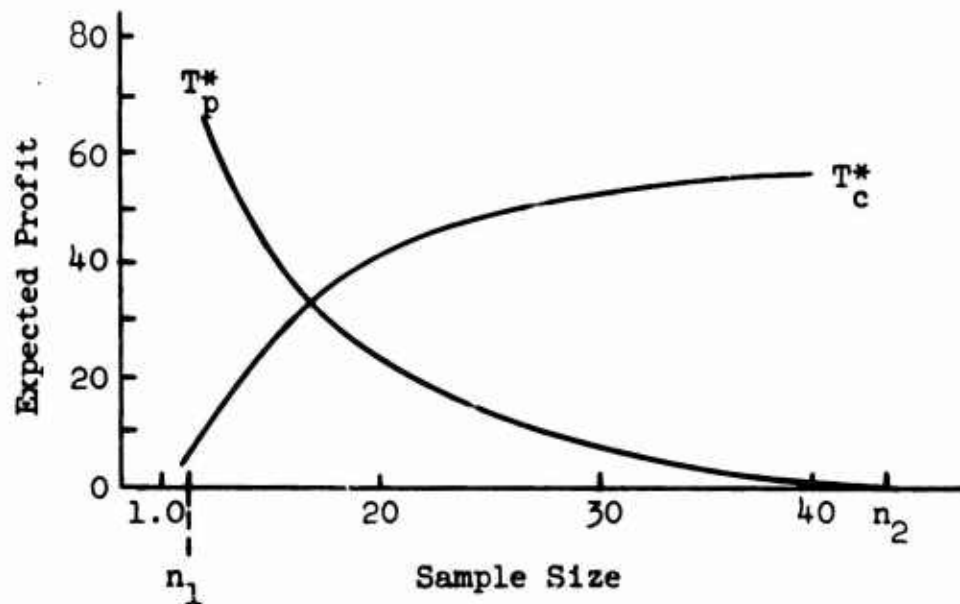| n	L	$T_p^* \times 100$	$T_c^* \times 100$	$T_c^* + T_p^* \times 100$	Probability of Producer receiving payment
10	1.099	76	-4.00	72.00	.904
11	1.005	67	4.60	71.60	.895
15	.766	43	27.00	70.00	.86
20	.602	26.50	41.50	68.00	.82
25	.508	16.50	49.50	66.00	.78
30	.444	10.00	54.00	64.00	.73
35	.404	5.30	56.70	62.00	.70
40	.368	1.80	58.20	60.00	.67
43	.354	0.00	58.80	58.80	.65
44	.351	-0.50	58.90	58.40	.64

Figure 9

We now prove that the special class of policies given by $\varphi(0) = L$, $\varphi(x) = 0$, $x > 0$, is admissible. By the definition of admissibility any admissible policy, (n,φ), must maximize T_p at p^*. Therefore consider

any policy, (n',φ'), not of the type given by (2.13), (2.16), (2.17), (2.18) that maximizes T_p at p^*, and assume that (n',φ') maps into (T_c',T_p') such that $T_c(p^*,\varphi',n') + T_p(p^*,\varphi',n') > T_c(p^*,L,n) + T_p(p^*,L,n)$. This is $V(p^*) - h(p^*) - c \cdot n' > V(p^*) - h(p^*) - c \cdot n$ and hence, $n' < n$. Since n_2 was chosen such that T_c^* increases in n for $n_1 \le n \le n_2$, we have $T_c(p^*,L,n') < T_c(p^*,L,n)$. By definition

$$T_c(p^*,L,\varphi_1,\ldots,\varphi_n,n') = NV(p^*) - c \cdot n' - \varphi(0)q^{*n'} - \sum_{x=1}^{n'} \varphi(x)\binom{n'}{x}p^{*x}q^{*n'-x} \ ,$$

and since the last term, $\sum_{x=1}^{n'} \varphi(x) \ \mathrm{pr}[x|n',p^*]$, is positive,

$$T_c(p^*,L,\varphi_1,\varphi_2,\ldots,\varphi_{n'},n') < T_c(p^*,L,n') \ .$$

Therefore

$$T_c(p^*,L,\varphi_1,\ldots,\varphi_{n'},n') < T_c(p^*,L,n)$$

which proves that (n,φ) satisfying (2.13), (2.16)-(2.18) is admissible.

Are there admissible strategies other than this class of policies that might yield a larger T_c? To answer this denote the special class by (n^*,φ^*) and assume first that n_2, the maximum sample size, is given by $T_p^* = 0$, as in the numerical example. Then if there existed an admissible policy (n',φ') such that $T_p' \ge 0$ and $T_c' > T_c^*$ we would have $T_c' + T_p' > T_c^* + T_p^*$ contradicting the admissibility of (n^*,φ^*).

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Alternatively assume that n_2 is given by $\Delta n T_c \geq 0$. Then if there were an admissible policy (n', φ') such that $T'_p \geq T^*_p$ and $T'_c > T^*_c$, we would have $T'_p + T'_c > T^*_p + T^*_c$ implying $T'_c < T^*_c$ which again is a contradiction.

Since n is a discrete variable, n_2 may be given by (2.17) with $T^*_p > 0$. In this case an additional positive payment, $\varphi(1)$, at the sample size $n_2 + 1$, might motivate the producer to p^*, yet increase the consumer's profit. This would depend on the cost of sampling. The paper reporting the Flehinger-Miller model uses an example in which the "sample size" is really a test length. Hence the test length may be varied continuously to drive the T^*_p exactly to zero.

We are left with two somewhat unsatisfying results. The Johns-Lieberman formulation implies that the consumer can both maximize his own profit and satisfy the producer with a sample size of one, and the Flehinger-Miller model derives optimal policies which are sampling plans with an acceptance number of one, except in special cases.

To understand these results consider the magnitudes of the actual prices and the confidence with which the results of these plans would be viewed. Consider first the implications of a lower bound on the prices in the Johns-Lieberman formulation. Let $\varphi(x) \geq m$ for all x, and let $n = 1$. From the conditions that both T_c and T_p be maximized at p^*, or $(1-p^*) \varphi_0 + p^* \varphi_1 = h(p^*)$, and $-\varphi_0 + \varphi_1 = h'(p^*)$ we derived the solutions $\varphi_0 = h(p^*) + p^*$, and $\varphi_1 = h(p^*) + p^* - 1$. If $h(p^*) + p^* - 1 < m$, we must choose $\varphi_1 = m$, and hence

$$q^*\varphi_0 + p^*\varphi_1 = q^*\varphi_0 + p^*K$$

$$> q^*[h(p^*) + p^*] + p^*[h(p^*) + p^* - 1]$$

$$= h(p^*) .$$

Thus the existence of a lower bound on the prices shows that in general $n = 1$ is not optimal, and the greater the lower bound, the greater will be the sample size necessary to allow the equations

$$g(p^*) = h(p^*) ,$$

$$g'(p^*) = h'(p^*)$$

to be satisfied simultaneously. The numerical results on page 58 are an instance in which $m = 0$, and $n = 43$ is the first sample size at which both $(.99)^n\varphi_0 = .23$ and $-n(.99)^{n-1}\varphi_0 = -10$ can be satisfied. Now bound the prices above by M, and again let $n = 1$. Then since we must have $-\varphi_0 + \varphi_1 = h'(p^*)$, and φ_i is bounded below by m, the minimum φ_0 is $\varphi_0 = m - h'(p^*)$. For large m or large negative $h'(p^*)$ it may occur that $m - h'(p^*) > M$. Again a larger sample size may be required so that $\varphi_0 \leq M.$[6]

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[6]Let $\pi_x = \binom{n}{x} p^x(1-p)^{n-x}$ and $dx = d/dp(\pi_x)$. Then since

$\sum\limits_{x=0}^{n} dx = 0, \ d_0 = -\sum\limits_{x=1}^{n} dx$. Now for $n > 1$, and $\varphi(x) = m, \ (x > 0)$,

$g'(p^*) = h'(p^*)$ becomes $-n\varphi_0 q^{*n-1} + m[nq^{*n-1}] = h'(p^*)$, or

$\varphi_0 = m - h'(p^*)/nq^{*n-1}$. For $p^* < 1/n+1, \ \Delta n[nq^{*n-1}] > 0$, and since

$h'(p^*) < 0, \ \varphi_0$ decreases in n.

Therefore the unbounded prices in the Johns-Lieberman model account for the optimality of small sample sizes. However the numerical results in Table 1 were certainly obtained with prices reasonably bounded relative to costs and values. Yet the sample size of 43 implies that the producer will receive a payment approximately twice in every three submitted batches, <u>even if submitted quality is constant at 1% defective</u>. In the terminology of acceptance sampling the producer is accepting a producer's risk of 35%! Also apparent is the fact that the consumer cannot specify an arbitrary degree of protection against quality poorer than p^*. A less arbitrary objection is that the sample size in the Flehinger-Miller model is determined only by the selected profit division. If, for instance, the producer were content to accept a 30% share of $V(p^*) - h(p^*)$ there would exist a unique n yielding an expected producer profit closest to this agreed on share.

The heart of the difficulty is that by using a policy with only a single payment when no defectives are observed in a sample of n, the sample size is a lever to force producer profit down,[7] rather than an instrument to assure both parties that the payment received and quality claimed by the producer are in fact correct. We feel that a contingent pricing policy should include both flexibility in the determination of bounded prices, and controllable protection for producer and consumer, and we develop and explore the implications and applications of such a model.

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[7] It is in this sense that Flehinger and Miller state that their sampling is non-informative but rather motivational.

CHAPTER III

CONSTRAINED CONTINGENT PRICING MODELS

Introduction

In Chapter II we noted that existing contingent pricing models
yield unrealistic policies due to the lack of bounds on the prices and
the lack of protection against sampling variation. We observed in
Chapter I that maximum expected profit and minimum total product cost
are not the sole concerns of producer and purchaser. Both consider risk,
and we distinguish here between two types of risk. When the firm accepts
a quality-contingent or fixed price contract it gambles on its cost
estimates. This is risk based on uncertainty in the production environ-
ment and is the risk generally discussed in economic theory. Risk of
this type is balanced by the minimum profit the producer considers
appropriate for the product. In a contingent pricing contract the pro-
ducer may actually achieve a specified quality but not receive the proper
payment due to variation in the number of defectives in randomly drawn
samples. This is statistical risk analgous to "producer's risk" in
acceptance sampling. The possibility of overpayment for poor lots
corresponds to "consumer's risk" in acceptance sampling. The existence
of statistical risk might cause the producer to negotiate for more profit,
but risk of this type is usually treated by designing plans that reduce
consumer and producer risk below acceptable limits.

Guthrie and Johns [46] developed an elegant model to determine the optimal sample size and acceptance number in single sampling plans without explicit consideration of consumer or producer risk, but they assumed the existence and knowledge of an underlying distribution of the random variable p. While the assumption that users can readily state their AQL, producer's risk, LTPD, and consumer's risk is an over-simplification, it would also be unusual to find a consumer confident in his estimate of the underlying prior distribution of the quality of an incoming product--especially a new product. The prior distribution is likely to be nonstationary, and the sampling plan ought to adapt as the distribution changes. But we have noted that the producer's attempts to change the distribution are themselves a function of the sampling plan. We will formulate a model using the concept of consumer and producer risk.

General Assumptions

We postulate the existence of a production and procurement situation with the following characteristics.

1. A single type of item is produced. Each item has a constant probability, p, of being defective. Batches of N are formed in such a way that the number of defectives, X, in each batch is a binomial random variable. Thus p is also the expected fraction defective per batch.

2. The production process is controlled by the producer who chooses p at a unit production cost, h(p), which we assume

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strictly convex and decreasing for p in some interval of interest.[1]

3. After delivery of the items the consumer draws a sample of n $(n \leq N)$ in such a way that the number of defectives in the sample, x, is a binomial random variable. The consumer pays $N\varphi(x)$ for the batch of N.

4. Sampling costs are paid by the consumer at c per unit inspected.

5. The consumer can tolerate some defective items in incoming lots and can state the expected value, $NV(p)$, of batches of N items containing X defectives where X is a random variable whose distribution is given by the binomial probability law with parameters N and p. We assume $V(p)$ concave.

6. The consumer desires to maximize expected net gain by concluding a contract promising quality which maximizes the difference between expected product worth and expected total procurement cost.

7. The consumer desires protection against overpayment when quality is poor. He can state the quality, p_b, he considers poor, and the maximum amount, Nv, he is willing to pay for quality as poor as p_b. Since any batch may yield a sample with

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[1] The basic unit production cost, say $\bar{h}(p)$, is known to the producer. During negotiation the producer adds a unit profit, say $z\bar{h}(p)$, and thus the existence of an $h(p) = (1+z)\bar{h}(p)$ becomes known to the consumer. The strict convexity is used in obtaining sufficient conditions for the existence of solutions.

very few defectives, the consumer can state the relative frequency, β, with which overpayment will be permissible.

8. The consumer will not agree to pay more than some amount, M.

9. Given a sample size, n, and payment schedule, $\varphi(x)$, the producer chooses p to maximize expected profit. Denote this p by p'.

10. The producer requires that the minimum price paid (or maximum penalty levied) be bounded below by m.

11. The producer requires that prices be monotonically decreasing in the number of defectives observed.

12. Since h(p) includes unit profit, the producer will agree to a contract only if at p' the expected payment is at least equal to h(p').

13. The producer desires protection against underpayment. He can state a minimum price, Nw, which the contingent pricing policy should assure him of receiving when quality is no worse than p'. Given that $p \leq p'$, payment less than Nw is permissible with frequency less than α.[2]

Notation

N = batch or lot size

X = number of defectives in the batch

n = sample size

x = number of defectives in the sample

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[2]We later relax these assumptions to discuss cases in which h(p) contains a random component, the producer's control over p is not precise, and the producer must replace all defectives discovered during inspection with nondefective items.

c = unit sampling cost

p = probability each item produced is defective

q = $(1-p)$

$NV(p)$ = expected value of a batch of N items when batches are formed according to Assumption 1

$N\varphi(x)$ = price paid for the batch of N when x defectives are observed

$h(p)$ = unit production cost to attain quality level p, including unit profit

p_b = quality level considered poor

Nv = maximum payment desirable when $p \geq p_b$

β = permissible frequency that $N\varphi(x)$ may be greater than Nv when $p \geq p_b$

p' = quality at which expected producer profit is maximized given n and $\varphi(x)$

Nw = minimum payment desirable when $p \leq p'$

α = permissible frequency that $N\varphi(x)$ may be less than Nw when $p \leq p'$

$g(p)$ = expected value of $\varphi(x)$ given p for fixed n

m = lower bound on possible payments, $\varphi(x)$

M = upper bound on possible payments, $\varphi(x)$

T_c = consumer expected profit

T_p = producer expected profit

p_o = quality which maximizes $V(p) - h(p)$

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$$\pi_x = \binom{n}{x} p^x (1-p)^{n-x} \quad ^3$$

$$d_x = d/dp \; \pi_x$$

Basic Model

The consumer and producer expected net gains are respectively

(3.1) $$T_c = NV(p) - Ng(p) - nc, \quad \text{and}$$

(3.2) $$T_p = Ng(p) - Nh(p).$$

Assumptions 1-13 imply that no contract is acceptable to both parties unless (3.3)-(3.6) are satisfied.

(3.3) $$pr[N\varphi(x) \geq Nw | p \leq p'] \geq 1 - \alpha$$

(3.4) $$pr[N\varphi(x) \leq Nv | p \geq p_b] \geq 1 - \beta$$

(3.5) $$M \geq \varphi(0) \geq \cdots \geq \varphi(n) \geq m$$

(3.6) $$Ng(p') - Nh(p') \geq 0$$

The consumer seeks to maximize (3.1) by choosing $n, \varphi(0), \ldots, \varphi(n)$[4] which satisfy (3.3)-(3.6) knowing that the producer will choose p to maximize (3.2).[5]

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[3] If it is necessary to stipulate the (n,p) at which π_x and d_x are defined we write $\pi_x(n,p)$ and $d_x(n,p)$.

[4] We also refer to $\varphi(0), \ldots, \varphi(n)$ as $\varphi_0, \ldots, \varphi_n$ and to $n, \varphi_0, \ldots, \varphi_n$ as (n,φ).

[5] In contrast to the Flehinger-Miller model, with this formulation of the problem, admissible policies are optimal for the consumer only if (3.6) holds with equality at $p = p_0$ for some n, and there is no smaller n yielding a higher consumer profit.

<u>Theorem 1</u>

For fixed n and p, say (n, p_0) where $0 < p_0 < 1$, it is neces-
sary that a set of prices, φ_X, maximizing (3.1) while simultaneously
maximizing (3.2) at p_0 and satisfying (3.3)-(3-6) be solutions to the
following linear programming program.[6]

(3.1')
$$\underset{\varphi}{\text{Minimize}} \sum_0^n \pi_X(n, p_0) \, \varphi_X$$

(3.2')
$$\sum_0^n d_X(n, p_0) \, \varphi_X = h'(p_0)$$

(3.3')
$$\varphi_{k_\alpha} \geq w$$

(3.4')
$$\varphi_{k_\beta} \leq v$$

(3.5')
$$M \geq \varphi_0 \geq \cdots \geq \varphi_n \geq m$$

(3.6')
$$\sum_0^n \pi_X(n, p_0) \, \varphi_X \geq h(p_0)$$

Proof:

By (3.1) maximizing T_c at (n, p_0) is equivalent to minimizing
$g(p_0)$ by choosing $\varphi_0, \ldots, \varphi_n$, and hence (3.1) is written as
(3.1'). A condition necessary for T_p to be maximized at $p_0 \epsilon(0, 1)$

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[6] k_α, k_β are described in the proof.

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is $\frac{d}{dp}[g(p) - h(p)] = 0$ at $p = p_o$, or $\sum_{x=0}^{n} d_x(n, p_o) \varphi_x = h'(p_o)$,

which is (3.2').[7] The constraints (3.5) and (3.6) are already linear

in φ_x.

The constraint (3.3) requires specifically

(3.7) $$pr[N\varphi(x) \geq Nw | p = p_o] \geq 1 - \alpha ,$$

which is equivalent to

(3.8) $$\sum_{k=0}^{n} pr[\varphi(k) \geq w] \ pr[x = k | p = p_o] \geq 1 - \alpha .$$

Let k'' be the largest integer for which $\varphi(k) \geq w$. From (3.8)

we must have $\sum_{k=0}^{k''} \pi_x(n, p_o) \geq 1 - \alpha$. Let k_α denote the smallest integer

such that $\sum_{k=0}^{k_\alpha} \pi_x(n, p_o) \geq 1 - \alpha$. In order for (3.8) to be satisfied,

it is necessary that $k'' \geq k_\alpha$, and by the monotonicity of $\varphi'(x)$, we see

that $\varphi(k_\alpha)$ cannot be less than w.

Thus (3.3) implies (3.3'). By similar reasoning (3.4) implies

(3.4'). Note also that (3.3') with k_α defined above implies (3.7).

We now show that (3.7) implies (3.3), and that (3.4') implies (3.4).

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[7]That $\varphi(x)$ satisfies (3.2') is not sufficient to insure that
$\varphi(x)$ satisfies (3.2). However, as will be seen from the remainder of
the proof, if a test of the points 0, 1, and all points for which
$g'(p) = h'(p)$ shows that T_p is maximized at p_o, then an optimal
solution to (3.1')-(3.6') is also an optimal solution to (3.1)-(3.6).
If T_p is maximized at $p_2 \neq p_o$, we can add the constraint
$T_p(p_o) > T_p(p_2)$ and determine a new set of prices.

For $j \leq n$, $\dfrac{d}{dp} \sum_o^j \pi_x(n,p) = \dfrac{-n!}{j!(n-j-1)!} p^j q^{n-j-1} \leq 0$. Hence $p < p_o$

implies $\sum_o^{k_\alpha} \pi_x(n,p) \geq \sum_o^{k_\alpha} \pi_x(n,p_o)$. Thus if (3.7) is satisfied, then (3.3) is satisfied.

Let k_β be the largest integer for which

$$(3.9) \qquad\qquad \sum_{k_\beta}^n \pi_x(n,p_b) \geq 1 - \beta \ ,$$

and require $\varphi_{k_\beta} \leq v$. We then have

$$pr[N\varphi(x) \leq Nv \mid p = p_b] = \sum_o^n pr[\varphi_k \leq v] \, pr[x = k \mid p_b]$$

$$\geq \sum_{k_\beta}^n pr[\varphi_k \leq v] \, pr[x = k \mid p_b] \geq 1 - \beta \ .$$

By reasoning as in the previous case we verify that (3.4') holding for k_β determined by (3.9) satisfies (3.4) for $p > p_b$. This completes the proof that solutions to (3.1)-(3.6) for fixed (n,p) are solutions to (3.1')-(3.6').

Variable Transformation

Before demonstrating a computational simplification in (3.1')-(3.6') we indicate the effect that (3.3), (3.4) have on the final prices and sample size. Let $\alpha = .10$, $\beta = .15$, $p_o = .05$, $p_b = .30$, $w = .50$, $v = .20$, and $n = 9$. Then $\sum_o^1 \pi_x(9,.05) = .928 > .9$, and $\sum_1^9 \pi_x(9,.3) = .96 > .85$. Thus $k_\alpha = 1$, $k_\beta = 1$ and (3.3'), (3.4') require $\varphi_1 \geq .5$ and $\varphi_1 \leq .2$, a contradiction. Therefore $n = 9$ permits no feasible prices for $p_o = .05$. At $n = 10$, $k_\alpha = 1$, $k_\beta = 2$,

and there may exist a feasible set of prices. If so, they will be constrained by $\varphi_1 \geq .5$, $\varphi_2 \leq .2$, indicating that both the minimum sample size and the form of the pricing policy are affected by (3.3) and (3.4).

The transformation

$$y_j = \varphi_j - \varphi_{j+1} \, , \qquad j = 0, 1, \ldots , n-1$$

(3.10)

$$y_n = \varphi_n - m$$

yields a linear programming problem equivalent to (3.1')-(3.6') and containing only 5 constraints. By (3.10), $\varphi_j = m + \sum_{i=j}^{n} y_i$, and choosing φ_x to minimize (3.1') is equivalent to choosing y_i to minimize

$$m + \sum_{i=0}^{n} \sum_{x=0}^{i} y_i \pi_x .$$

Let $P_o^i = \sum_o^i \pi_x(n, p_o)$, and $D_o^i = \sum_o^i d_x(n, p_o)$. The transformed problem follows.

(3.1")
$$\underset{y_i}{\text{Minimize}} \quad \sum_{i=0}^{n} P_o^i y_i$$

(3.2")
$$\sum_{i=0}^{n} D_o^i y_i = h'(p_o)$$

(3.3")
$$\sum_{i=k_\alpha}^{n} y_i \geq w - m$$

(3.4")
$$\sum_{i=k_\beta}^{n} y_i \leq v - m$$

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$$(3.5") \qquad\qquad \sum_{i=0}^{n} y_i \leq M - m$$

$$(3.6") \qquad\qquad \sum_{i=0}^{n} P_o^i y_i \geq h(p_o) - m$$

Since this problem need have $y_i \neq 0$ for at most five y_i, there need be at most six distinct price levels, φ_x, one of which may be m. If any of the constraints (3.3")-(3.5") are redundant the number of different price levels will be correspondingly reduced. If, for example, $\alpha = 1$, $\beta = 1$, $M = L$, and $m = -L$, where L is positive and very large, we obtain the price schedules characterized in Chapter II, since the constraint (3.5") will not be active if $M - m$ is very large, and the constraints (3.3") and (3.4") can be satisfied by considering only y_o and y_n. We can select y_o, y_n to satisfy (3.2")-(3.4"), and (3.6"), setting all other $y_i = 0$. The price schedule is then

$$\varphi_o = m + y_o + y_n$$

$$\varphi_x = m + y_n , \qquad\qquad (x = 1, \ldots, n) .$$

The number of active constraints, and thus the number of distinct price levels, φ_x, depend on the parameters, α, β, M, m, w, and v.

Policies Optimal at p_o

Every point (n,p') is either infeasible for (3.2')-(3.6') or yields a value of (3.1') and a set of φ_x with the property that (n,φ) induces the manufacturer to select p'. Let R denote the set of (n,p) for which φ_x exist satisfying (3.1')-(3.6'), and let $G(n,p)$

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denote the value of (3.1') for $(n,p)\epsilon R$. We consider conditions on

M, m, w, and v under which there exists a policy (n,φ) maximizing

T_p at p_o, and yielding $G(n,p_o) = h(p_o)$. Assume $V(p)$, $h(p)$, α, β,

and p_b fixed. There are twelve possible cases, determined by the

relations $w \underset{>}{<} v$, and the order relations among $[np]$, k_α, and k_β.

The most restrictive cases are those for which $w > v$. We will consider

one case with $w > v$ and one with $w \leq v$.

Proposition 1

Let $\quad D_a^b = \sum_a^b d_x(n,p_o)$

$$P_a^b = \sum_a^b \pi_x(n,p_o) \text{ , and}$$

$$s = [np] .$$

Case I:

Assume $w \leq v$ and $k_\alpha < s < k_\beta$. If

$$(3.11) \qquad\qquad (M-m) \geq \frac{h'(p_o)}{D_o^s} , \qquad \text{and}$$

$$(3.12) \qquad m = [h(p_o) - w] + \frac{h'(p_o)}{(M-m)\ D_o^s} [w - (M-m)\ P_o^s]$$

then $G(n,p_o) = h(p_o)$.

Proof:

Note first that $w \leq v$ implies that φ_i satisfying (3.3) and

(3.4) can be found for all $n \geq 1$. For $\sum_o^{k_\alpha} \pi_x(n,p_o) \geq 1 - \alpha$ can always

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be satisfied by $k_\alpha = n$ and $\sum_{k_\beta}^{n} \pi_x(n, p_b) \geq 1 - \beta$ can be satisfied

with $k_\beta = 0$. In this event we have $v \geq \varphi_0 \geq \varphi_n \geq w$ which does not

violate (3.5).

Therefore, for any n we have $\max_\varphi \sum_0^n d_i \varphi_i = w \sum_0^n d_i = 0$ for

φ_i satisfying (3.3)-(3.5). Also, $\min_\varphi \sum_0^n d_i \varphi_i = M D_0^s + m D_{s+1}^n = (M-m) D_0^s$.

By (3.11) $(M-m) \geq h'(p_0)/D_0^s$, hence $(M-m) D_0^s \leq h'(p_0)$, and we obtain

$$(3.13) \qquad \min_\varphi \sum d_i \varphi_i \leq h'(p_0) < \max_\varphi \sum d_i \varphi_i \; .^{[8]}$$

Therefore there exist φ_x satisfying (3.2)-(3.5). Assume that

$\min_\varphi \sum d_i \varphi_i < h'(p_0)$. Then for $0 < \lambda < 1$,

$$h'(p_0) = \lambda(M-m) D_0^s + (1-\lambda) w D_0^n = \lambda(M-m) D_0^s \; , \quad \text{and}$$

$$(3.14) \qquad \lambda = \frac{h'(p_0)}{(M-m) D_0^s} \; .$$

We now show that $\qquad \varphi_x = \lambda M + (1-\lambda) w \; , \quad 0 \leq x \leq s$

$$\varphi_x = \lambda m + (1-\lambda) w \; , \quad s \leq x \leq n$$

yield equality in (3.6). By (3.12)

$$m = h(p_0) - w + \frac{h'(p_0)}{(M-m) D_0^s} [w - (M-m) P_0^s] \; , \text{ or by } (3.14)$$

$$m = h(p_0) - w + \lambda[w - (M-m) P_0^s], \text{ which yields}$$

$$h(p_0) = (\lambda M + (1-\lambda) w) P_0^s + (\lambda m + (1-\lambda) w) P_{s+1}^n = G(n, p_0).$$

[8] The strict inequality holds by the original assumption that $h(p)$ is strictly convex and decreasing.

Case II:

Assume $w > v$, $k_\alpha < s < k_\beta$, and assume that for some $n_0 \leq N$, $k_\alpha(p_0, n) < k_\beta(p_b, n)$. Then if

$$(3.15) \qquad w - v \leq \frac{h'(p_0)}{D_0^{k_\beta - 1}(n_0, p_0)}$$

$$(3.16) \qquad M - m \geq \frac{h'(p_0)}{D_0^s(n_0, p_0)}$$

$$(3.17) \quad \frac{h(p_0) - m - (M-m) P_0^s}{(w-v) P_0^{k_\beta - 1} + (v-m) - (M-m) P_0^s} = \frac{h'(p_0) - (M-m) D_0^s}{(w-v) D_0^{k_\beta - 1} - (M-m) D_0^s}$$

Then $G(n_0, p_0) = h(p_0)$.

Proof: At $n = n_0$, $\max_\varphi \sum d_i \varphi_i = w D_0^{k_\beta - 1} + v D_{k_\beta}^n = (w-v) D_0^{k_\beta - 1}$.

By (3.15) $(w-v) D_0^{k_\beta - 1} \geq h'(p_0)$, and hence $\max_\varphi \sum d_i \varphi_i \geq h'(p_0)$. At

$n = n_0$, $\min_\varphi \sum d_i \varphi_i = M D_0^s + m D_{s+1}^n = (M-m) D_0^s$. By (3.16) $(M-m) D_0^s \leq h'(p)$,

and hence $\min_\varphi \sum d_i \varphi_i \leq h'(p_0)$. Thus there exist φ_i satisfying (3.2)-(3.5). In the most general case

$$h'(p_0) = \lambda[(w-v) D_0^{k_\beta - 1}] + (1-\lambda)[(M-m) D_0^s]$$

yielding $\qquad \lambda = \dfrac{h'(p_0) - (M-m) D_0^s}{(w-v) D_0^{k_\beta - 1} - (M-m) D_0^s}$

By (3.17)

$$\frac{h(p_0) - m - (M-m) P_0^S}{(w-v) P_0^{k_\beta-1} + (v-m) - (M-m) P_0^f} = \frac{h'(p_0) - (M-m) D_0^S}{(w-v) D_0^{k_\beta-1} - (M-m) D_0^S} = \lambda$$

From which

$$h(p_0) = \lambda\left[wP_0^{k_\beta-1} + v(1-P_0^{k_\beta-1})\right] + (1-\lambda)\left[MP_0^S + M(1-P_0^S)\right]$$

or

$$h(p_0) = \lambda\left[wP_0^{k_\beta-1} + vP_{k_\beta}^n\right] + (1-\lambda)\left[MP_0^S + mP_{s+1}^n\right],$$

and (3.6) holds with equality.

When either of these sets of conditions is satisfied, we need only compute p_0, n_0, and generate the optimal prices, $\varphi(x)$, by solving $(3.1')-(3.6')$.

Admissible Policies

Recall that in Chapter II we required of an admissible policy that it maximize joirt profit, $\psi(n,p) = NV(p) - Nh(p) - nc$. Under our assumptions $\psi(n,p)$ is concave in n and p, and hence if the set R is convex in n for fixed p and convex in p for fixed n, we may determine (n_0,p_0) by any one of several algorithms or search methods. We will later describe the Fibonacci search generalized to two variables and under the above assumptions this method will determine (n_0,p_0). To see this, assume as proven that the Fibonacci search procedure will determine $p_0(n)$ to maximize $\psi(n,p)$ for fixed n. We will show that

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$\psi(n, p_o(n))$ is concave in n. By the concavity of $\psi(n, p)$ and the definition of $p_o(n)$,

$$\lambda \psi(n_1, p_o(n_1)) + (1-\lambda) \psi(n_2, p_o(n_2))$$

$$\leq \psi(\lambda_1 n_1 + (1-\lambda) n_2, \lambda p_o(n_1) + (1-\lambda) p_o(n_2))$$

$$\leq \psi(\lambda n_1 + (1-\lambda) n_2, p_o(\lambda n_1 + (1-\lambda) n_2)) .$$

Thus the convexity of R allows efficient determination of the unique (n_o, p_o).

Let $\underline{n}$ = minimum n for which $p = 0$ satisfies (3.3), (3.4).

$\overline{n}$ = minimum n for which $p = p_o$ satisfies (3.3), (3.4).

If R is convex in p for fixed n but not in n for fixed p, we must evaluate $\psi(n, p_o(n))$ for every $n \epsilon [\underline{n}, \overline{n}]$ to determine (n_o, p_o). There is no feasible $n < \underline{n}$, and for $n > \overline{n} \quad \psi(\overline{n}, p_o) \geq \psi(n, p_o) \geq \psi(n, p_o(n))$.

We state as a proposition the conditions under which R is convex in p for fixed n. The proof is similar to the proof of proposition 1.

Proposition 2

Let $\overline{p}(n)$ = the greatest p for which $\sum_o^{k_\beta - 1} \pi_x(n, p) \geq 1 - \alpha$ where k_β is the greatest integer for which $\sum_{k_\beta}^n \pi_x(n, p_b) \geq 1 - \beta$. Let p_1 be that p at which $h'(p) = L$, where L is a specified constant. Let p_2 be the minimum p at which $V(p) = h(p)$. Let $\underline{p} = \max(p_1, p_2)$. Let the maximum $\sum d_i \varphi_i$ satisfying (4.3')-(4.5') be minimized at p', and the minimum $\sum d_i \varphi_i$ be maximized at p'', where p', p'' are in the interval $[\underline{p}, \overline{p}(n)]$.

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Assume $\underline{p} \leq \bar{p}(n)$, $w > v$, and $[np] < k_\alpha < k_\beta$. Then if

$$(3.18) \qquad (w-m) \leq \frac{h'(\bar{p}(n))}{D_o^{k_\beta - 1}(p')} \quad ,$$

$$(3.19) \qquad (M-m) \geq \frac{1}{D_o^s(p'')}\left[h'(\underline{p}) - \frac{h'(\bar{p}(n)) \; D_{s+1}^{k_\alpha}(p'')}{D_o^{k_\beta - 1}(p')}\right], \quad \text{and}$$

$$(3.20) \qquad h(p) \leq \lambda\left[wP_o^{k_\beta - 1}(p') + vP_{k_\beta}^n(p')\right]$$

$$+ (1-\lambda)\left[MP_o^s(p'') + wP_{s+1}^{k_\alpha}(p'') + mP_{k_\alpha + 1}^n(p'')\right]$$

are satisfied where λ is given by (3.21) then R is convex in p for fixed n, for all $p\epsilon[\underline{p},\bar{p}(n)]$.

Proof:

Abbreviate $\bar{p}(n)$ by $\bar{p}$. By definition of $\underline{p}$, $\bar{p}$, (3.3) and (3.4) hold for $p\epsilon[\underline{p},\bar{p}]$.

Since $h''(p) > 0$, $\max_p h'(p) = h'(\bar{p})$, and $\min_p h'(p) = h'(\underline{p})$.

$\underset{\varphi}{\text{Max}} \sum d_i\varphi_i = (w-v) \; D_o^{k_\beta - 1}(p)$, and $\underset{\varphi}{\min} \sum d_i\varphi_i = (M-m) \; D_o^s(p) + (w-m)D_{s+1}^{k_\alpha}(p)$.

Therefore by (3.18) $\underset{\varphi}{\max} \sum d_i\varphi_i \geq (w-v) \; D_o^{k_\beta - 1}(p') \geq h'(\bar{p})$. By (3.19)

$$D_o^s(p'')(M-m) + \frac{h'(\bar{p}) \; D_{s+1}^{k_\alpha}(p'')}{D_o^{k_\beta - 1}(p')} \leq h'(\underline{p}), \text{ and by } (3.18)$$

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$D_o^s(p'')(M-m) + (w-m) \, D_{s+1}^{k_\alpha}(p'') \leq h'(p)$, or $\min\limits_\varphi \sum d_i \varphi_i \leq h'(p)$. Hence for

for all $p\epsilon[\underline{p}, \bar{p}]$ there exist φ_x satisfying (3.2)-(3.5). For some

$p\epsilon[\underline{p}, \bar{p}]$ let

$$h'(p) = \lambda \left[(w-v) \, D_o^{k_\beta - 1}(p') \right] + (1-\lambda) \left[(M-m) \, D_o^s(p'') + (w-m) \, D_{s+1}^n(p'') \right]$$

yielding

$$(3.21) \qquad \lambda = \frac{h'(p) - (M-m) \, D_o^s(p'') + (w-m) \, D_{s+1}^{k_\alpha}(p'')}{(w-v) \, D_c^{k_\beta - 1}(p') - (M-m) \, D_o^s(p'') - (w-m) \, D_{s+1}^{k_\alpha}(p'')} \ .$$

By (3.20), (3.6) is satisfied, and therefore R is convex in p for

fixed n for $p\epsilon[\underline{p}, \bar{p}]$.

Conditions (3.18)-(3.20) are not restrictive and frequently equality

holds in (3.20). When this is so we may determine $p_o(n)$ by a search

procedure. Convexity in n for fixed p is obtained only in small

intervals $[n_1, n_2]$. Heuristically this is due to the changes in k_α

and k_β as n increases. For fixed (n_1, p_1), k_α can be less than

k_β, while at $n_1 + 1$ k_α may increase while k_β remains constant,

producing infeasibility at (n_1+1,p). At $n_1 + 2$ k_β may increase,

again creating a feasible point, (n_1+2,p). Thus (3.3) and (3.4) can

cause nonconvexity in n for fixed p.

R will be convex in n for fixed p and $n\epsilon[n_1, n_2]$ when either

$w \leq v$, or $F_x(k_\beta(n) -1) \geq 1 - \alpha$. We have shown in proposition 1 that

for $w \leq v$ any (n,p) satisfies (3.3') and (3.4') with $k_\alpha = n$ and

$k_\beta = 0$. For $w > v$, $F_x(k_\beta(n) - 1) \geq 1 - \alpha$ is easily satisfied for

large α or large p_b. For if in some interval $[n_1, n_2]$ $k_\beta(n)$ is constant then as n increases $F_x(k_\alpha)$ decreases, but for large α $F_x(k_\alpha) \geq 1 - \alpha$ can hold, and for large p_b, $F_x(k_\beta-1) \geq 1 - \alpha$ can hold. Of course, if as n increases, $k_\beta(n)$ increases, then $F_x(k_\beta(n) - 1) \geq 1 - \alpha$ can hold.

In the constrained maximization of $\psi(n,p)$ it will generally be possible to use a search or programming algorithm to determine $p_o(n)$, but a search method will not always determine n_o. In the latter case $\psi(n,p_o(n))$ must be evaluated at every $n \in [\underline{n}, \overline{n}]$.

Maximization of $T_c(n,p)$

Assume that by search or iteration we have determined $\psi(n_o,p_o)$. This is an upper bound on $T_c(n,p)$ since $g(p) \geq h(p)$ implies $G(n,p) \geq h(p)$, or

$$T_c(n,p) = NV(p) - NG(n,p) - nc \leq NV(p) - Nh(p) - nc = \psi(n,p) .$$

The upper bound can be attained when equality holds in (3.20) at (n_o,p_o). In maximizing $T_c(n,p)$ any (n,p) yielding $\psi(n_o,p_o)$ is a global optimum. The conditions stated in Proposition 2 and the assumption that the interval of interest is convex in n insure only that $G(n,p)$ is defined on some regular region. Without further restrictions on $G(n,p)$ we cannot insure that $T_c(n^*,p^*)$ determined by any search or nonlinear programming method is globally optimal. We briefly summarize the structure of R and then investigate the behavior of $G(n,p)$ in n and p.

Assume that the conditions of Proposition 2 are satisfied and that we need only remark on constraints (3.3) and (3.4). There is a minimum

n, $\underline{n}$, in R, such that for $n \geq \underline{n}$ $p = 0$ satisfies (3.3) and (3.4), and for each $n\epsilon R$ there is a maximum p, $\bar{p}(n)$ satisfying (3.3), (3.4). Although this $\bar{p}(n)$ is not generally monotonically increasing in n, there may be intervals $[n_1, n_2]$ in which R is convex in n and p. Proposition 3 states conditions under which $G(n,p)$ decreases in n on such intervals.

Lemma 1

If $k_\alpha > [np]$ and $k_\alpha(n+1) = k_\alpha(n)$, $\Delta n \sum_o^{k_\alpha} d_x \leq 0$

Proof:

$$\Delta n \sum_o^{k_\alpha} d_x = \frac{-(n+1-k_\alpha)}{q} \pi_{k_\alpha}(n+1,p) + \frac{(n-k_\alpha)}{q} \pi_{k_\alpha}(n,p)$$

$$= \frac{\pi_{k_\alpha}(n,p)}{q} [(np-k_\alpha) + (p-1)]$$

$$= (-1) \frac{\pi_{k_\alpha}(n,p)}{q} [(k_\alpha-np) + (1-p)] \leq 0 .$$

Proposition 3

Let $\quad P_1 = \sum_o^{k_\alpha} \pi_x(n,p) \qquad\qquad D_1 = \sum_o^{k_\alpha} d_x(n,p)$

$\qquad\quad P_2 = \sum_o^{k_\alpha} \pi_x(n+1,p) \qquad\qquad D_2 = \sum_o^{k_\alpha} d_x(n+1,p)$

Assume that for (n,p), $(n+1,p)$, we have

$$(3.22) \qquad G(n,p) = L_1 \sum_o^{k_\alpha(n)} \pi_x(n,P) + J_1 \sum_{k_\alpha(n)+1}^{n} \pi_x(n,p) \quad ,$$

(3.23) $G(n+1,p) - L_2 \sum_o^{k_\alpha(n+1)} \pi_x(n+1,p) + J_2 \sum_{k_\alpha(n+1)+1}^{n+1} \pi_x(n+1,p)$.

Let $k_\alpha(n) = k_\alpha(n+1)$, $k_\alpha > [np]$, and $\epsilon = h'(p)\left[\dfrac{P_1}{D_1} - \dfrac{P_2}{D_2}\right]$.

Then if $J_2 \leq J_1 + \epsilon$, $G(n+1,p) \leq G(n,p)$.

Proof:

By (3.2') $L_1 D_1 + J_1(-D_1) = h'(p)$ or $(L_1-J_1) = \dfrac{h'(p)}{D_1}$, and

similarly $(L_2-J_2) = \dfrac{h'(p)}{D_2}$. Therefore by Lemma 1 $(L_2-J_2) \leq (L_1-J_1)$.

Since $0 \leq P_2 \leq P_1$ we have $(L_2-J_2) P_2 \leq (L_1-J_1) P_1$ or

$h'(p)\left[\dfrac{P_1}{D_1} - \dfrac{P_2}{D_2}\right] \geq 0$. By hypothesis, $J_2 \leq J_1 + h'(p)\left[\dfrac{P_1}{D_1} - \dfrac{P_2}{D_2}\right]$ and we

obtain $J_2 + (L_2-J_2) P_2 \leq J_1 + (L_1-J_1) P_1$, which by (3.22), (3.23) is

$G(n+1,p) \leq G(n,p)$.

Behavior of $G(n,p)$ in p

The behavior of $G(n,p)$ in p depends on the changes in

$\dfrac{\sum_o^J \pi_x(n,p)}{\sum_o^J d_x(n,p)}$ relative to changes in $\dfrac{h(p)}{h'(p)}$ as p varies. If at some

p_o, $G(n,p_o) = h(p_o)$ and for $p < p_o$ $h''(p) >> 0$, then as p decreases

$h'(p)$ becomes negative rapidly. If in the same interval $\sum_o^J d_x(n,p)$

becomes more negative less rapidly so that $\varphi_x(x \leq j)$ increases in order

that $\sum d_i \varphi_i = h'(p)$, then $\sum \pi_i \varphi_i$ will increase and may yield

$\sum \pi_i \varphi_i > h(p)$ for $p < p_o$. In Proposition 4 we indicate conditions for

which $G(n,p) - h(p)$ increases as p decreases.

Lemma 2

If $k_\alpha > [np]$, and $k_\alpha(p)$ is constant in some interval I, then

$$\frac{d}{dp} \sum_0^{k_\alpha} d_x(n,p) \leq 0 \quad \text{for} \quad p \in I.$$

Proof:

Let $p_1, p_2 \in I$, $p_1 < p_2$, and $k_\alpha(p_1) = k_\alpha(p_2)$. Then

$$\sum_0^{k_\alpha} d_x(n,p) = \frac{-n!}{(k_\alpha)!(n-k_\alpha-1)!} p^{k_\alpha} q^{n-k_\alpha-1} = \frac{(n-k_\alpha)}{q} \pi_{k_\alpha}(n,p) .$$

Since $d_x(n,p) > 0$ for $x > [np]$, $\pi_{k_\alpha}(n,p_2) > \pi_{k_\alpha}(n,p_1)$. Therefore

$$\frac{\pi_{k_\alpha}(n,p_2)}{q_2} > \frac{\pi_{k_\alpha}(n,p_1)}{q_1} \quad \text{and} \quad \frac{-(n-k_\alpha)}{q_2} \pi_{k_\alpha}(n,p_2) < \frac{-(n-k_\alpha)}{q_1} \pi_{k_\alpha}(n,p_1) \quad \text{or}$$

$$\sum_0^{k_\alpha} d_x(n,p_2) \leq \sum_0^{k_\alpha} d_x(n,p_1).$$

Proposition 4

Let I be an interval in which the conclusions of Lemma 2 are valid and $\frac{d}{dp} \sum_0^{k_\alpha} d_x(n,p) \leq 0$. For $p_1 < p_2$ let $G(n,p)$ be given by

$$(3.24) \qquad G(n,p_1) = L_1 \sum_0^{k_\alpha} \pi_x(n,p_1) + J_1$$

$$(3.25) \qquad G(n,p_2) = L_2 \sum_0^{k_\alpha} \pi_x(n,p_2) + J_2$$

Let $\quad P_1 = \sum_0^{k_\alpha} \pi_x(n,p_1) \qquad\qquad D_1 = \sum_0^{k_\alpha} d_x(n,p_1)$

$$P_2 = \sum_0^{k_\alpha} \pi_x(n,p_2) \qquad\qquad D_2 = \sum_0^{k_\alpha} d_x(n,p_2)$$

Then if $J_1 \geq J_2$ and $\begin{vmatrix} h(p_1) & P_1 \\ h'(p_1) & D_1 \end{vmatrix} > \begin{vmatrix} h(p_2) & P_2 \\ h'(p_2) & D_2 \end{vmatrix}$,

$G(n,p_1) - h(p_1) \geq G(n,p_2) - h(p_2)$.

Proof:

By hypothesis $D_2 h(p_2) - P_2 h'(p_2) < D_1 h(p_1) - P_1 h'(p_1)$, and by

Lemma 2 $D_2 < D_1$ or $-D_2 > -D_1$. Therefore

$$\frac{D_2 h(p_2) - P_2 h'(p_2)}{D_2} > \frac{D_1 h(p_1) - P_1 h'(p_1)}{D_1}$$

or $h(p_2) - h'(p_2) \frac{P_2}{D_2} > h(p_1) - h'(p_1) \frac{P_1}{D_1}$.

By $(3.2')$ $\frac{h'(p_2)}{D_2} = L_2 - J_2$, and $\frac{h'(p_1)}{D_1} = L_1 - J_1$. Thus

$(L_2-J_2) P_2 - h(p_2) < (L_1-J_1) P_1 - h(p_1)$. By hypothesis $J_1 \geq J_2$ and

therefore $G(n,p_2) - h(p_2) < G(n,p_1) - h(p_1)$ completing the proof.

Renegotiating Parameters

We have been concerned with stating the conditions under which

(n,p) can be found to maximize the consumer profit, $T_c(n,p)$. Beyond

stating conditions that yield $G(n,p) = h(p)$ in some region, there is

little we can say about $G(n,p)$ that aids in negotiation of the param-

eters M, m, w, v, α, β. Of course as the constraints $(3.3)-(3.6)$

are relaxed the set of points on which $G(n,p) = h(p)$ increases, but

conversely as the constraints become increasingly restrictive the

resultant $G(n,p)$ may become defined on a collection of unconnected

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sets. In this event we could resort to a random search procedure to determine (n^*, p^*),[9] but it seems artificial in light of the original procurement situation to stop with such a solution.

The existence of a solution $(n^*, p^*) \neq (n_o, p_o)$ and such that $T_c(n^*, p^*) < \psi(n_o, p_o)$ can occur in two ways. The discrimination required between p_o and p_b may be so fine that it is more profitable for the consumer to insist on higher quality to allow discrimination at a smaller sample size. Or the bounds w and m may be so high that in order to motivate production at p_o, the expected payment is greater than $h(p_o)$. Even if (n^*, p^*) is correct for fixed parameters, the consumer might ask how far he would have to relax M, v, or β, to obtain a policy closer to (n_o, p_o). Whether such relaxation were worth the incremental profit would depend on the consumer's risk preferences. The consumer might also find it necessary and profitable to offer the producer a higher unit profit in return for agreement on decreased m, α, and w.

In the following example a minimum price of 60 per batch yielded a consumer profit of 350.60, and a minimum price of -51.75 per batch

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[9] In Brooks' random search method we would choose, say, 30 points (n, p) at random in the region R, and evaluate $T_c(n, p)$ at each point. Assume we are interested in the event that the greatest T_c thus obtained actually ranks among the top 10% of all points (n, p) in R. The probability of this event is $1 - (1-.1)^{30}$ or $.958$. This method makes no assumptions about the concavity of $T_c(n, p)$ or the convexity of R. Further refinement could be obtained by choosing points about the maxima obtained, or initiating a gradient procedure at the highest points obtained by random search. [47]

yielded a profit of 351.65. The consumer might offer the producer additional profit up to 1.05 per batch in return for the decreased lower bound.

Let $N = 500$, $\alpha = .1$, $w = h(p)$, $M = 1$, $c = .2$, $p_b = .2$,

$\beta = .4$, $v = .3986$, $m = .12$, $V(p) = (1-p)$, and

$h(p) = .154 - .22p + .066e^{.02/p}$

The solution with $m = .12$ is found at $n^* = 18$ and $p^* = .0528$.

$\varphi_0 = .2918$

$\varphi_1 = .2171$

$\varphi_2 = .2171$

$\varphi_3 = .12$

$\vdots$

$\varphi_{18} = .12$
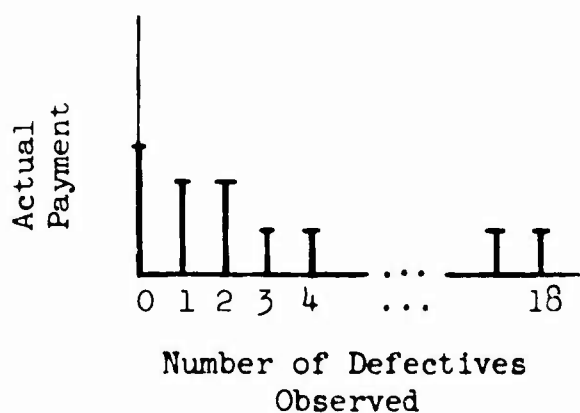

Figure 1

The solution with $m = -.1035$ is found at $n^* = 13$ and $p^* = .0528$.

$\varphi_0 = .2788$

$\varphi_1 = .2178$

$\varphi_2 = .2178$

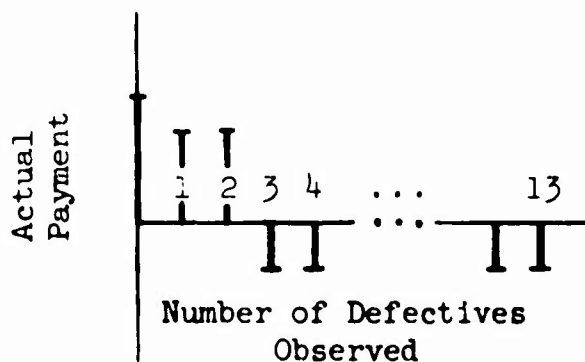$\varphi_3 = -.1035$

$\vdots$

$\varphi_{13} = -.1035$

Figure 2

While the original intent in developing this basic model was the introduction of bounds on prices and assignable protection against sampling variation, we do not insist that parameter values be stated only once, but suggest that during negotiation the parameters of the model will be successively adjusted and the unit profit varied, until when the contract is concluded, $(n*, p*) = (n_o, p_o)$.

Computational Procedure

In order to successively adjust parameters, we must know the value of $T_c(n*, p*)$ and $\psi(n_o, p_o)$. In experimenting with representative data we found the conditions of Proposition 2 (convexity of R in p for fixed n, with $G(n,p) = h(p)$) satisfied for a wide range of parameters. We found R convex in n for fixed p less often, and concluded that while the $T_c(n,p)$ surface had too much structure to justify a random search, it did not have enough for a gradient method. We therefore used a generalization of the Fibonacci search technique.

The Fibonacci search in a single variable is fully developed by Wilde [48], and computational experience with the extension to more than one variable is treated by Krolak and Cooper.[10] [49] The generalization

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[10]A Fibonacci search in a single variable using n search points can be described as follows. The Fibonacci sequence is $f_o = 1$, $f_1 = 1$, $f_n = f_{n-1} + f_{n-2}$, $(n \geq 2)$. For unimodal $\psi(p)$ defined on $a_1 \leq p \leq b_1$, let $I_1 = b_1 - a_1$ and define Δ_2 to be $(f_{n-2}/f_n) I_1$. Let $p_1 = \Delta_2 + a_1$, $p_2 = b_1 - \Delta_2$. Then if $\psi(p_1) \geq \psi(p_2)$, $a_2 \Leftarrow a_1$, $b_2 \Leftarrow p_2$, and $p_3 = a_2 + \Delta_3$ where $\Delta_3 = (f_{n-3}/f_{n-1}) I_2$. If $\psi(p_2) > \psi(p_1)$ then $a_2 \Leftarrow p_1$, $b_2 \Leftarrow b_1$, and $p_3 \Leftarrow b_2 - \Delta_3$. This continues until the maximum lies in an interval of length $I_n = I_1(f_o/f_n)$.

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to two variables requires that when searching in the variable n,
$T_c(n,p)$ is evaluated at $p*(n)$, where $p*(n)$ maximizes $T_c(n,p)$ for
fixed n. In order to avoid determining the region R, we modify
constraints $(3.3')$-$(3.5')$ by introducing artificial variables as follows.

$$(3.3^o) \qquad \qquad \varphi_{k_\alpha} + \theta_1 \geq w$$

$$(3.4^o) \qquad \qquad \varphi_{k_\beta} + \theta_2 \leq v$$

$$(3.5^o) \qquad \qquad \begin{cases} \varphi_o + \theta_3 \leq M \\ \\ \varphi_n + \theta_4 \geq m \end{cases}$$

The θ_i are unrestricted in sign, and we change $(3.1')$ to

$$(3.1^o) \qquad \underset{\varphi, \theta}{\text{Minimize}} \quad \sum_o^n \pi_i \varphi_i + Q \sum_1^4 |\theta_i|$$

where Q is a very large positive number. This problem will be feasible
at all points of some conveniently defined region, and points feasible
for the original problem $(3.1')$-$(3.6')$ will be feasible for the modified
problem with all $\theta_i = 0$.

The generalized Fibonacci search is computationally efficient.[11]
Furthermore it provides a grid of the points evaluated which serves as a

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[11]The procedure was programmed in Fortran II for the IBM 7090 at
Stanford University. The number of points evaluated is optional, and no
attempt was made to optimize the program. Using a ten point search over
the sample sizes and a ten point search over the interval $(0, p_b)$ the
procedure may be viewed as 100 sequential evaluations to determine the
maximum of a function on a 143×89 lattice. Solutions almost always
require less than 30 seconds.

check both for multimodality and for an irregular search region. If examination of the points evaluated suggests that the search did not terminate with an optimal solution, a more detailed evaluation may be necessary, and is possible by redefining the search region. The computational procedure can be succinctly written as

$$\text{Max}_{n} \ \text{Max}_{p} \ \left\{ NV(p) - nc - \text{Min}_{\varphi} \sum \pi_i(n,p) \ \varphi_i \right\}$$

We actually increase efficiency by proceeding as follows:

1. Find p_o to maximize $V(p) - h(p)$.

2. After specifying a lower bound and an arbitrary number of search points, begin Fibonacci search in the variable n.

3. At each n proceed directly to p_o and evaluate $G(n,p_o)$ by solving the linear programming problem (3.1")-(3.6"), as modified in (3.1°) and (3.3°)-(3.5°).

   a. If $G(n,p_o) = h(p_o)$, $p*(n) = p_o$.

   b. If $G(n,p_o) > h(p_o)$, begin a Fibonacci search in the variable p to determine $p*(n)$.

For cases in which $G(n,p)$ is unimodal in p and decreasing in n, or in which $G(n,p_o) = h(p_o)$ for some interval $[n_1, n_2]$, this procedure terminates with an optimal (n,p). This will be the case when the conditions of Proposition 1 are satisfied on $[n_1, n_2]$. Otherwise the procedure provides no assurance that the global maximum will be attained, but due to the possible nonconvexity of R and nonconcavity of $T_c(n,p)$, no procedure short of evaluating every sample size in $[\underline{n}, \overline{n}]$ can promise a global maximum with certainty.

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The following example illustrates points discussed during development of the basic model.

Example 1

Let $\quad V(p) = 1 - 3p, \quad p \leq p_b$

$$= 0 \qquad p > p_b$$

$$h(p) = .4 - .2p - .01e^{.05/p} \; ,$$

$$c = .172, \quad N = 200, \quad M = 1, \quad m = .15 \; ,$$

$$p_b = .175, \quad v = .45V(p) \; , \quad \beta = .10 \; ,$$

$$w = .75h(p), \quad \alpha = .05$$

The optimal contingent pricing policy is found at $\quad n^* = 39$, with $p^* = .0325$, and is shown in Figure 3.

$$\varphi_x = .5243 \quad x = 0, 1$$

$$\varphi_x = .3580 \quad x = 2, 3$$

$$\varphi_x = .2137 \quad x = 4, \ldots, 7$$
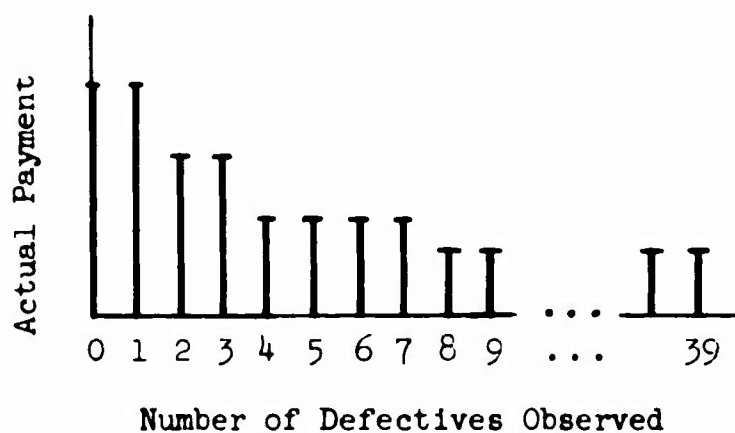
$$\varphi_x = .15 \quad x = 8, \ldots, 39$$

Number of Defectives Observed

Figure 3

In Figure 4 and Table 1, the expected payment and profit resulting from the policy (n, φ) is displayed.

Figure 4

Table 1

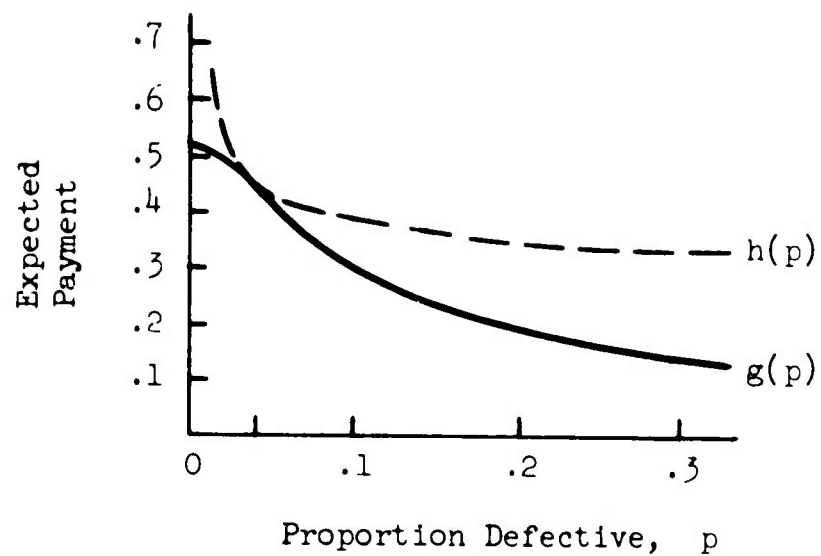| p	Value at p	Cost at p	Expected Payment at p	Producer Profit at p	Consumer Profit at p
.01ɔ	.921	2.4758	0.514	-1.961	0.4069
.015	.906	0.7894	0.504	-0.285	0.4019
.020	.891	0.5666	0.492	-0.074	0.3988
.025	.876	0.4984	0.479	-0.019	0.3969
.030	.861	0.4681	0.465	-0.002	0.3957
.0325	.849	0.4544	0.454	0.000	0.3951
.035	.846	0.4514	0.451	-0.000	0.3950
.040	.831	0.4409	0.436	-0.004	0.3944
.045	.816	C.4335	0.422	-0.011	0.3937
.050	.801	0.4281	0.408	-0.019	0.3928

In Figure 5 a grid of $T_c(n,p)$ in the region surrounding $(n*,p*)$ is shown, we denote infeasibility by *.

Sample Size	.0225	.0250	.0275	.0300	.0325	.0350	.0375
41	*	.3123	.3839	.3906	.3925	*	*
40	*	.3074	.3804	.3919	.3938	*	*
39	*	.3022	.3767	.3931	.3950	.3948	*
38	*	*	*	*	*	*	*

Proportion Defective, p

Figure 5

Figure 6 indicates the increase in $T_c(n^*,p^*)$ with increasing α.
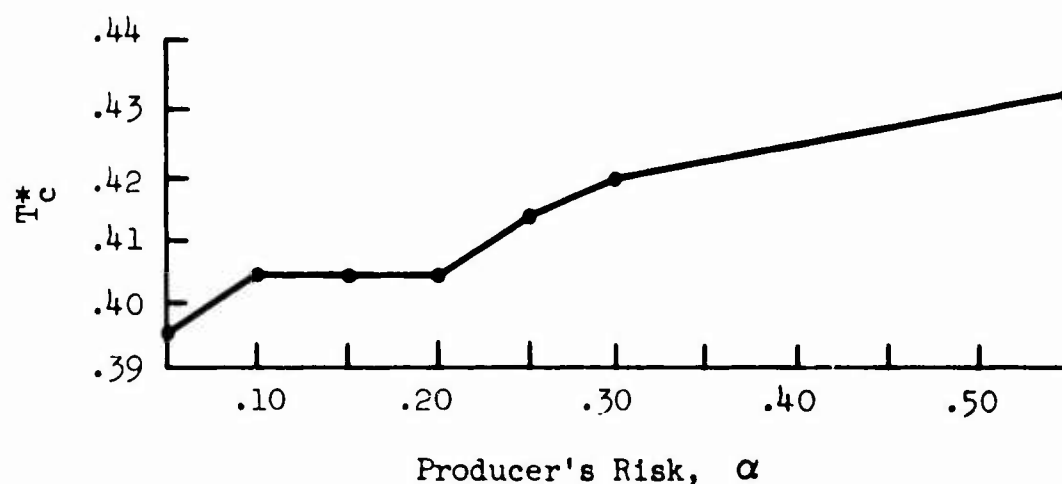

Figure 6

Piecewise Linear Price Schedules

Existing incentive price schedules are usually piecewise linear. In the simplest and most usual case incentive payments are determined by negotiating a sample size, maximum and minimum payments, and the number of defectives at which the maximum and minimum payment occur. The linear pricing policy shown in Figure 7 can be written as follows.

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$$\varphi_x = \varphi_o \qquad\qquad\qquad\qquad\qquad 0 \leq x \leq a$$

$$\varphi_x = \varphi_o \left[\frac{b - x}{b - a} \right] + \varphi_n \left[\frac{x - a}{b - a} \right] \qquad a < x < b$$

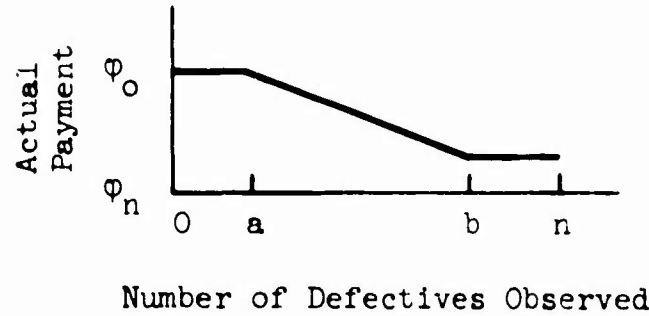$$\varphi_x = \varphi_n \qquad\qquad\qquad\qquad\qquad b \leq x \leq n$$

Figure 7

We will modify the basic model to derive linear policies maximizing consumer net gain subject to the general assumptions 1-13. Assume that the consumer chooses the sample size, n, and the pricing policy, (a, b, $\varphi(x)$) to maximize (3.1) subject to (3.3), (3.4), (3.6), and (3.26), knowing that the producer will choose p' to maximize (3.2).

$$(3.26) \qquad \begin{cases} \varphi_o \leq M \\[4pt] \varphi_x = \varphi_o & x = 0, 1, \ldots, a \\[4pt] \varphi_x = \varphi_o \left[\dfrac{b - x}{b - a} \right] + \varphi_n \left[\dfrac{x - a}{b - a} \right] & x = a + 1, \ldots, b - 1 \\[4pt] \varphi_x = \varphi_n & x = b, \ldots, n \\[4pt] \varphi_n \geq m \end{cases}$$

For fixed n, p_o, $(0 < p_o < 1)$, a set of prices, φ_x, maximizing (3.1) while simultaneously maximizing (3.2) at p_o and satisfying (3.3), (3.4), (3.6), and (3.26) are solutions to the linear programming problem (3.1')-(3.6') with (3.5') replaced by (3.26). The justification for

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this statement is identical to the proof of Theorem 1 and is not repeated.

The transformation (3.10) reduces the linear programming problem to one containing only 6 constraints.

(3.27)
$$\underset{y_i}{\text{Minimize}} \sum_a^{b-1} P_o^i y_i$$

(3.28)
$$\sum_a^{b-1} D_o^i y_i = h'(p_o)$$

(3.29)
$$\sum_{\max[a,k_\alpha]}^{n} y_i \geq w - m$$

(3.30)
$$\sum_{\max[a,k_\beta]}^{b-1} y_i \leq v - m$$

(3.31)
$$\begin{cases} \sum_a^{b-1} y_i \leq M - m \\ \left[\dfrac{b-a}{b-x}\right] \sum_x^{b-1} y_i = \sum_a^{b-1} y_i \end{cases}$$

(3.32)
$$\sum_a^{b-1} P_o^i y_i \geq h(p_o) - m$$

We use the following notation in discussing linear pricing policies.

(3.33)
$$P_o = \sum_o^a \pi_x + \sum_{a+1}^{b-1} \left(\frac{b-x}{b-a}\right) \pi_x$$

(3.34)
$$P_n = \sum_b^n \pi_x + \sum_{a+1}^{b-1} \left(\frac{x-a}{b-a}\right) \pi_x$$

(3.35)
$$D_o = \sum_o^a d_x + \sum_{a+1}^{b-1} \left(\frac{b-x}{a+1}\right) d_x$$

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$$(3.36) \qquad D_n = \sum_b^n d_x + \sum_{a+1}^{b-1} \left(\frac{x-a}{b-a} \right) d_x \ . \,^{[12]}$$

Let p_o maximize $V(p) - h(p)$. For fixed a and b, there do not in general exist φ_o and φ_n that maximize T_c at p_o. Relations (3.1) and (3.2) can be written

$$(3.37) \qquad P_o \varphi_o + P_n \varphi_n \geq h(p_o) \qquad \text{and}$$

$$(3.38) \qquad D_o \varphi_o + D_n \varphi_n = h'(p_o) \ .$$

Assume that with equality in both equations, the solution is φ_o^o, φ_n^o. If $\varphi_n^o < m$, the solution is not feasible by (3.26), and hence $\sum \pi_i \varphi_i > h(p_o)$. Therefore the possibility again exists of an optimal solution at $p^* \neq p_o$, and as in the basic case we search for $p^*(n)$ and n^* with a generalized Fibonacci search procedure. For fixed (n,p) we find $a^*(n,p)$, and $b^*(a,n,p)$ by iteration after first restricting the number of points to be evaluated. We will develop the reduction procedure after presenting an example of a linear pricing policy derived by this method.

Example 2

Let $\quad V(p) = 1 - 1.5p \ , \quad p \leq .17$

$\qquad\qquad = 0 \qquad\qquad p > .17$

$\quad h(p) = .25 - .415p - .06 \ln p$

$\quad c = .12, \quad N = 500, \quad M = .45, \quad m = .10,$

$\quad p_b = .17, \quad v = .4V(p), \quad \beta = .25,$

$\quad w = h(p), \quad \alpha = .25.$

---

[12] If $b = a + 1$ the second summation is deleted.

The optimal linear pricing policy is found at $n^* = 46$, $p^* = .052$, with $a = 3$, $b = 7$, $\varphi_0 = .4209$, and $\varphi_{46} = .11$. This is shown in Figure 8.

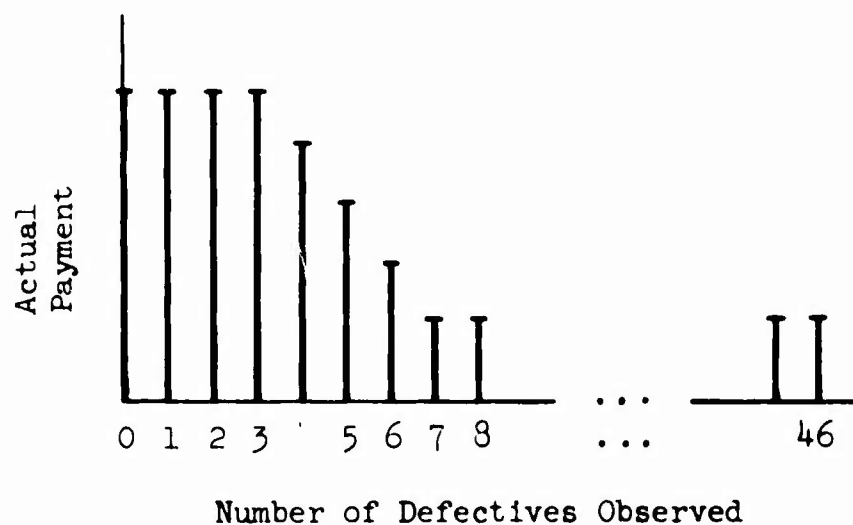Number of Defectives Observed

Figure 8

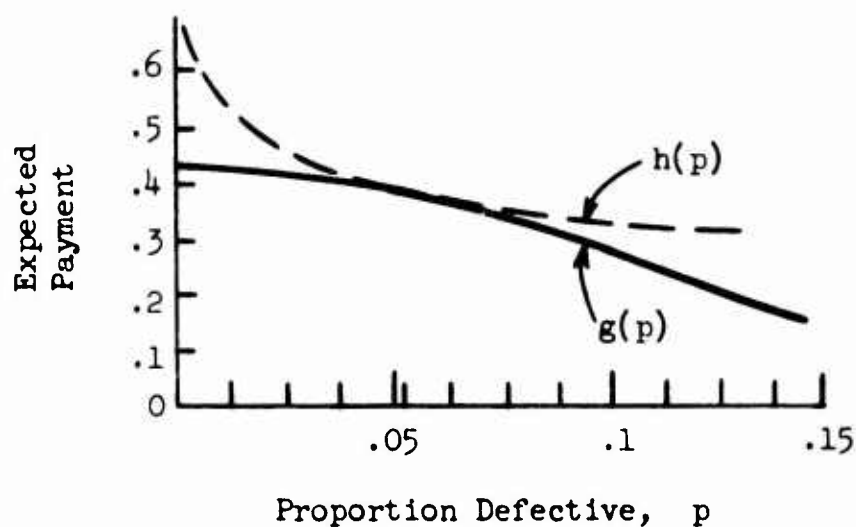In Figure 9 and Table 2 we indicate the expected payments, costs and profits resulting from this policy.

Figure 9

Table 2

p	Value at p	Cost at p	Expected Payment at p	Producer Profit at p	Consumer Profit at p
.010	.9740	.5037	.4208	-.0830	.5532
.020	.9590	.4608	.4196	-.0411	.5393
.030	.9440	.4339	.4159	-.0180	.5280
.040	.9290	.4137	.4085	-.0052	.5205
.050	.9140	.3970	.3968	-.0002	.5171
.052	.9110	.3940	.3940	.0000	.5170
.060	.8990	.3827	.3810	-.0016	.5179
.070	.8840	.3699	.3617	-.0082	.5223
.080	.8690	.3582	.3397	-.0186	.5293
.090	.8540	.3475	.3160	-.0315	.5380
.100	.8390	.3374	.2916	-.0458	.5473

Determining Feasible (a,b)

For fixed n and p a may lie in the interval $[0, n-1]$, and b in the interval $[a+1, n]$, yielding $n(n+1)/2$ points. We use constraints (3.3) and (3.4) to initially reduce the number of points. Assume that $w > v$, and that n is sufficiently large so that $p = 0$ satisfies (3.3) and (3.4), and let $\bar{p}(n)$ be the greatest p satisfying (3.3), (3.4). Therefore for all $p\epsilon[0, \bar{p}(n)]$, $k_\alpha(p) < k_\beta$. Now if $a \geq k_\beta$, we would have $v \geq \varphi_{k_\beta} = \varphi_{k_\alpha} \geq w$, a contradiction. Thus we need never consider $a > k_\beta - 1$. Similarly if $a < k_\alpha$ then if $b < k_\alpha$ we would have $w \leq \varphi_{k_\alpha} = \varphi_b = \varphi_{k_\beta} \leq v$ with a contradiction as before. Therefore for $a < k_\alpha$ we need only consider $b > k_\alpha$.

For fixed a we obtain bounds on feasible b. For $a < k_\alpha$ when $\varphi_a = M$, we must have $\varphi_{k_\alpha} \geq w$. Let $s = \frac{M - w}{a - k_\alpha}$. Then b_{min} is given

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by the smallest integer greater than or equal to $\frac{v - w}{s} + k_\alpha$, as shown in Figure 10. If $\varphi_{k_\alpha} = w$, b cannot exceed some b_{max} which would
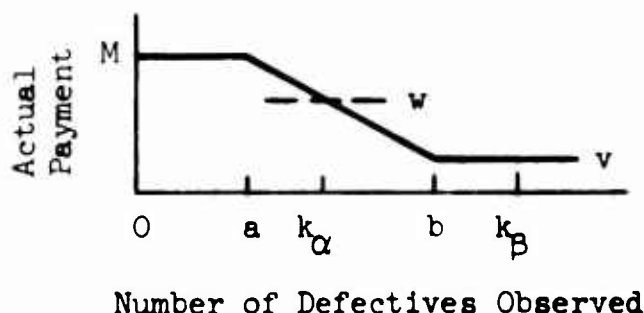
Figure 10

yield $\varphi_{k_\beta} > v$ as shown in Figure 11. Therefore b_{max} is given by the greatest integer less than $k_\beta + \frac{m - v}{s}$, where $s = \frac{v - w}{k_\beta - k_\alpha}$.
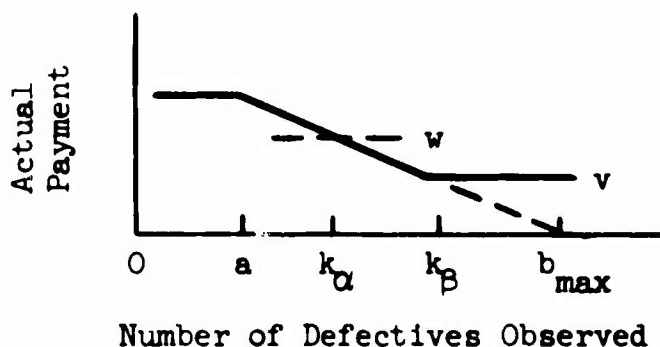
Figure 11

If $b_{max} < b_{min}$ there exist no prices for the current a' that satisfy (3.2')-(3.4'), (3.26), and (3.6'). Furthermore all $a < a'$ are infeasible since b_{max} is constant in a but b_{min} increases as a decreases. For $k_\alpha \le a \le k_\beta - 1$ we see $b_{min} = a + 1$. Because $\varphi_a \ge w$ and $\varphi_{k_\beta} \le v$, b_{max} is the greatest integer less than $a + \frac{m - w}{s}$, where $s = \frac{w - v}{a - k_\beta}$. See Figure 12.

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Number of Defectives Observed
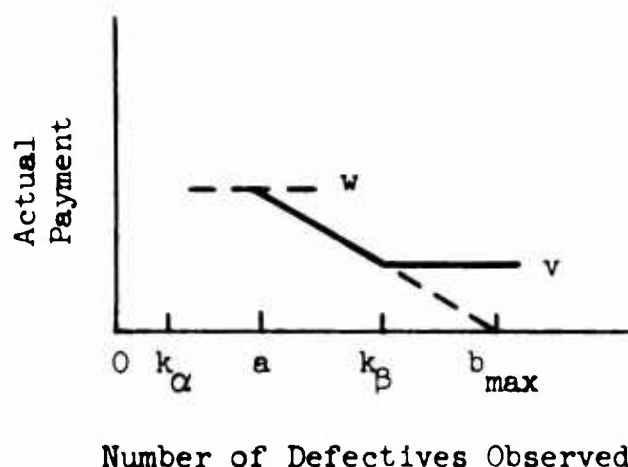
Figure 12

Thus rather than $n(n+1)/2$ points we consider only $a \in [0, k_\beta - 1]$, $b \in [b_{min}, b_{max}]$, and if $b_{max} < b_{min}$ evaluation terminates in b. We further reduce the number of points evaluated by terminating iteration in b when $T_c(b)$ first decreases in b, and terminate iterating in a when $T_c(a)$ first decreases in a. This procedure is appropriate when $T_c(a,b)$ is unimodal in a and b. If $T_c(a,b)$ is not unimodal in either variable this terminating rule is certain of attaining only a local maximum. If the value of $T_c(a,b)$ at the local maximum is equal to $\psi(n,p)$, then this maximum is global over (a,b), regardless of the unimodality of $T_c(a,b)$.

The relatively few (a,b) remaining after the reduction process makes numerical determination of linear pricing policies practicable. Computation time is longer than in the basic model due to the search for a* and b* for each (n,p). Solutions for a "typical" problem involving 50 search points require about 40 seconds on the IBM 7090. The most efficient procedure is to solve the problem with the basic model

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and use the resulting sample size as a lower bound in the linear model, reducing the number of sample sizes to be searched.

Unimodality of $T_c(a,b)$

We state the conditions in Proposition 5 under which $G(n,p,a,b) = h(p)$ for fixed parameters M, m, w, v. Alternatively these are the conditions on the parameters so that $G(n,p,a,b) = h(p)$ for fixed a, b. The proof is simply a restatement of $(3.1')$-$(3.6')$ and will be omitted.

Proposition 5

Let $\displaystyle \mathcal{D} = \begin{vmatrix} P_o & P_n \\ D_o & D_n \end{vmatrix}$, $\displaystyle \mathcal{D}_o = \begin{vmatrix} h(p) & P_n \\ h'(p) & D_n \end{vmatrix}$, $\displaystyle \mathcal{D}_n = \begin{vmatrix} P_o & h(p) \\ D_o & h'(p) \end{vmatrix}$.

Then $G(n,p,a,b) = h(p)$ if and only if (3.39)-(3.42) hold.

$$(3.39) \qquad \frac{\mathcal{D}_n}{\mathcal{D}} \geq m$$

$$(3.40) \qquad \frac{\mathcal{D}_o}{\mathcal{D}} \leq M$$

$$(3.41) \qquad \mathcal{D}_o[b - k_\alpha] + \mathcal{D}_n[k_u - a] \geq w\mathcal{D}(b-a)$$

$$(3.42) \qquad \mathcal{D}_o[b - k_\beta] + \mathcal{D}_n[k_\beta - a] \leq v\mathcal{D}(b-a) .$$

Whenever $G(n,p) \equiv h(p)$ for (a,b) in some $A \times B$, and the problem is infeasible outside that region, the iterative procedure will always select the smallest a and largest b. It is possible for $T_c(a,b)$ to be unimodal in a and b, and in the next two

propositions we indicate conditions sufficient for $T_c(a,b)$ to be unimodal in a. Analgous conditions can be derived for b.

Lemma 3

Let Δ_a represent the first difference in a and let P_o, P_n, D_o, D_n be as defined in (3.33)-(3.36). Then $\Delta_a P_o > 0$, $\Delta_a P_n < 0$, and if $a + 1 > np$ $\Delta_a D_o > 0$ and $\Delta_a D_n < 0$. This statement is also true if Δ_a is replaced by Δ_b.

Proof:

$$d_x = \binom{n}{x} p^{x-1}(1-p)^{n-x-1}(x-np).$$ Therefore if $a + 1 \geq np$ then $d_{a+1} \geq 0$.

$$\Delta_a D_o = d_{a+1}\left[\frac{1}{b-a}\right] + \sum_{a+2}^{b-1} d_x\left[\frac{b-x}{(a-b)(a-b+1)}\right] \quad .$$

Both terms are positive. Therefore $\Delta_a D_o > 0$. The remaining statements of the lemma follow in the same manner by writing out the definitions of the quantities.

Proposition 6

Denote the dependence of φ_o, P_o, D_o, φ_n, P_n, D_n on a as φ_o^a, P_o^a, etc., and abbreviate Δ_a by Δ. If for some $a' > np + 1$, $\varphi_o^{a'} = w$, and if for $np + 1 \leq a < a'$ the solution, φ_o', to (3.37), (3.38) is increasing in a, then $G(a-1) > G(a)$ for $np + 1 \leq a < a'$.

Proof:

By hypothesis $\varphi_o^{a'} = w$. Either the solution to the equations (3.37), (3.38) yields w, or the solution yields $\varphi_o' < w$ and $\varphi_o^{a'} = w$ by (3.3') and (3.26). In either case since the solution, φ_o', is

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increasing in a, $\varphi_0^{a'-1} = w$. Therefore, for every a in the interval of interest, $\varphi_0^a = w$.

At a we have $h'(p) = wD_0^a + \varphi_n^a D_n^a$, and an analgous expression at a - 1. Therefore

$$\varphi_n^{a-1} = \frac{w\Delta D_0 + \varphi_n^a D_n^a}{D_n^{a-1}} \quad , \qquad \text{and}$$

$$G(a-1) = wP_0^{a-1} + \frac{P_n^{a-1}}{D_n^{a-1}} \left[w\Delta D_0 + \varphi_n^a D_n^a \right] \quad , \qquad \text{from which}$$

$$G(a-1) - G(a) = P_n^{a-1} \left\{ w\left[\frac{\Delta P_0}{P_n^{a-1}} + \frac{\Delta D_0}{D_n^{a-1}} \right] + \varphi_n^{a-1} \left[\frac{D_n^a}{D_n^{a-1}} - \frac{P_n^a}{P_n^{a-1}} \right] \right\} \quad .$$

We will show the quantity in brackets is positive, proving the proposition. By (3.4') $\varphi_n^{a-1} \leq w$, and it suffices therefore to show

$\frac{D_n^a}{D_n^{a-1}} - \frac{P_n^a}{P_n^{a-1}} \leq \frac{\Delta P_0}{P_n^{a-1}} + \frac{\Delta D_0}{D_n^{a-1}}$. This is equivalent to

$\frac{\Delta D_0 - D_n^a}{D_n^{a-1}} \geq \frac{-\Delta P_0 - P_n^a}{P_n^{a-1}}$, which is $\frac{D_0^a - D_0^{a-1} - D_n^a}{D_n^{a-1}} \geq -1$, or $\Delta_a D_0 \geq \Delta_a D_n$.

But by Lemma 3 $\Delta_a D_0 > 0$ and $\Delta_a D_n < 0$.

Proposition 7

Assume $p \neq 0$ or 1, and for some $a'' \geq np + 1$, $\varphi_n^{a''} = m$. Assume also that for $a > a''$ the solution, φ_n', to the equations (3.37), (3.38) decreases in a. Then $G(a) < G(a+1)$ for $a > a''$.

Proof:

Case I: $m \leq 0$. By (3.2) $\varphi_o^a D_o^a + m D_n^a = \varphi_o^{a+1} D_o^{a+1} + m D_n^{a+1}$ and

$$\varphi_o^a = \frac{\varphi_o^{a+1} D_o^{a+1}}{D_o^a} + \frac{m \Delta D_n}{D_o^a}. \quad \text{From which}$$

$$(3.43) \qquad \frac{\Delta G(a)}{P_o^a} = \varphi_o^{a+1} \left[\frac{P_o^{a+1}}{P_o^a} - \frac{D_o^{a+1}}{D_o^a} \right] - m \left[\frac{\Delta D_n}{D_o^a} - \frac{\Delta P_n}{P_o^a} \right].$$

By Lemma 3, $\Delta_a P_o > 0$, and thus $\dfrac{P_o^{a+1}}{P_o^a} > 1$. $\Delta_a D_o > 0$ and $D_o^{a+1} < 0$

imply $\dfrac{D_o^{a+1}}{D_o^a} < 1$. Therefore the quantity in the first bracket is positive.

By Lemma 3 $\Delta D_n < 0$, and $\Delta P_n < 0$. Since $D_o^a < 0$, and $P_o^a > 0$, the

quantity $\left\{ -m \left[\dfrac{\Delta D_n}{D_o^a} - \dfrac{\Delta P_n}{P_o^a} \right] \right\}$ is positive. Therefore $\Delta G(a)$ is positive.

Case II: $m > 0$. From (3.43) since $\varphi_o > m$ in any case of interest,

it suffices to show $\dfrac{P_o^{a+1}}{P_o^a} - \dfrac{D_o^{a+1}}{D_o^a} \geq \dfrac{\Delta D_n}{D_o^a} - \dfrac{\Delta P_n}{P_o^a}$. This statement is

$$\frac{P_o^{a+1} + \Delta P_n}{P_o^a} \geq \frac{D_o^{a+1} + \Delta D_n}{D_o^a} \quad \text{which is} \quad \frac{P_o^a}{P_o^a} \geq \frac{D_o^a}{D_o^a}, \quad \text{proving Case II.}$$

If the conditions of Propositions 6 and 7 are satisfied, and if

$a' \leq a''$ with $T_c(a)$ constant in a when $a' < a < a''$, then $T_c(a)$ is

unimodal in a.

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Once a* and b* are determined for fixed n and p, the problem
of determining n* and p* is identical to that in the basic case,
except that the linear model is more restrictive. With the contingent
pricing problem formulated in this manner, there is no advantage in
restricting attention to linear policies. The same parameters must be
negotiated in both cases, and the linear model must always require more
computational effort and can never result in a higher consumer profit
than the basic model.

For the data in Example 2 the basic model yielded the following
pricing policy at $n* = 36$ and $p* = .0522$.

$$\varphi_x = .4399 , \quad x = 0, 1$$

$$\varphi_x = .3933 , \quad x = 2, 3$$

$$\varphi_x = .2285 , \quad x = 4, \ldots , 8$$

$$\varphi_x = .1100 , \quad x = 9, \ldots , 36$$

This policy is displayed in Figure 13, and the linear policy is displayed
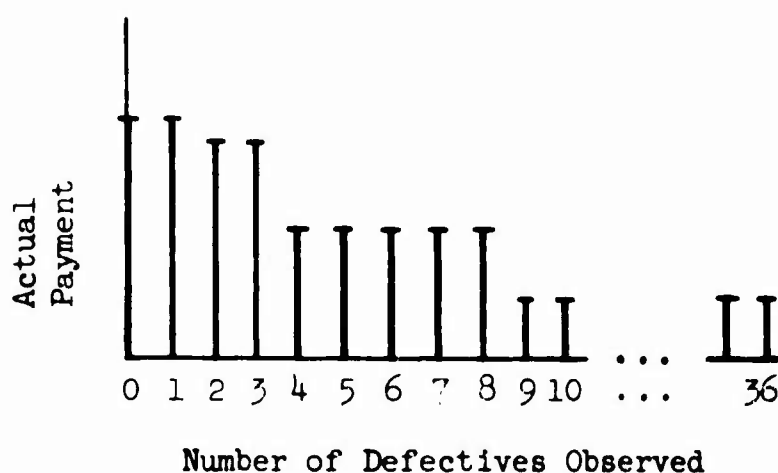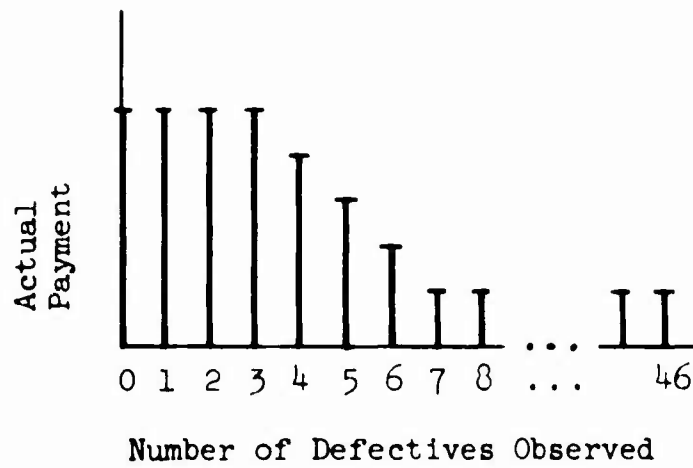in Figure 14.

Figure 13

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Figure 14

The expected payments are displayed in Figure 15 for both policies, and
Table 3 contains the expected cost and profits for the nonlinear policy.
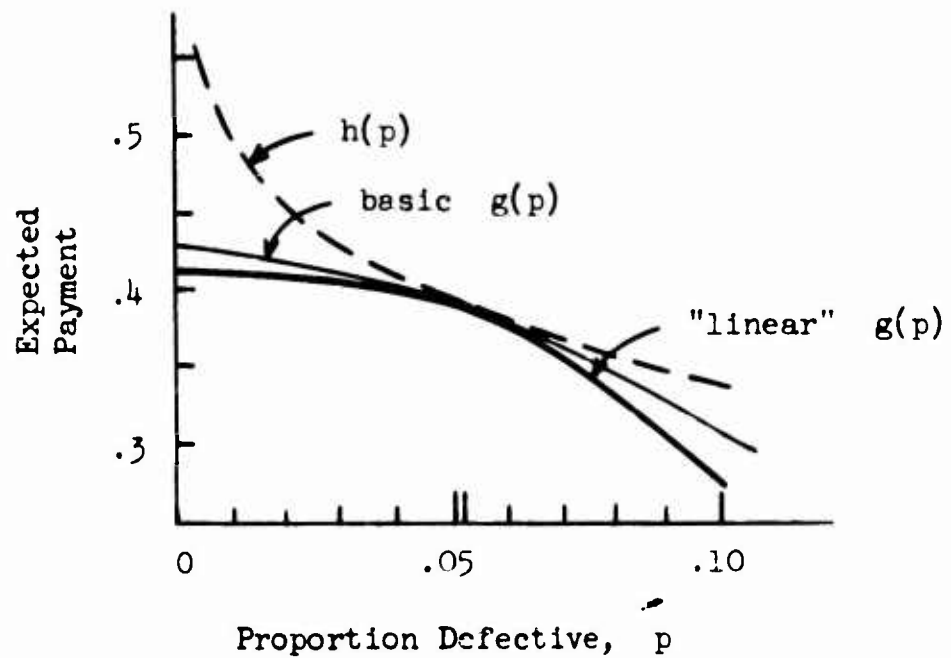
Figure 15

Table 3

p	Cost at p	Expected Payment at p	Producer Profit at p	Consumer Profit at p
.010	.5037	.4374	-0.0663	.5389
.020	.4608	.4312	-0.0295	.5301
.030	.4339	.4222	-0.0117	.5242
.040	.4137	.4106	-0.0031	.5208
.050	.3970	.3969	-0.0001	.5194
.052	.3936	.3936	.0000	.5194
.055	.3896	.3894	-0.0001	.5194
.060	.3827	.3817	-0.0009	.5196
.070	.3699	.3656	-0.0042	.5207
.080	.3582	.3493	-0.0089	.5220
.090	.3475	.3333	-0.0142	.5231
.100	.3374	.3178	-0.0196	.5235
.170	.2787	.2328	-0.0459	.5036

Consumer unit expected profit using the linear policy is .5170, and using the less restricted price schedule derived by the basic model the expected unit profit is .5194. From Figure 15 it can be noted that the policy at sample size 36 does not provide the same discrimination in expected payment that the policy at $n = 46$ does, but both policies satisfy the conditions (3.3) and (3.4). In situations where the producer prefers a linear price schedule due to ease of interpretation, the consumer must decide whether to offer increased expected profit in return for producer acceptance of the non-linear price schedule.

Extensions and Applications

We indicate situations in which the basic assumptions 1-13 can be weakened and a more general class of production and procurement situations considered.

Replacement of Discovered Defectives

In most instances in which inspection or testing is non-destructive, a producer must replace all defectives discovered during the inspection process at no additional cost to the consumer. We will modify the basic model to treat this situation and exhibit a pricing policy derived from the data used in Example 1.

Assume that the cost to the producer of replacing discovered defectives is the production cost, $h(p)$, plus a fixed charge, r, reflecting additional transportation, testing, and special handling costs. The expected net gains are

$$(3.44) \qquad T_c(n,p) = (N-n) \, V(p) + n[V(0) - c] - Ng(p) \, ,$$

$$(3.45) \qquad T_p(n,p) = Ng(p) - Nh(p) - npr \, .$$

We will make the constraint (3.3) more realistic by providing the producer some assurance of regaining a portion of expected cost. Therefore the consumer chooses (n,φ) to maximize (3.44) knowing that the producer will choose p' to maximize (3.45), and such that (n,φ) satisfy

$$(3.46) \qquad pr[N\varphi(x) \geq Nw(h(p') + rx) \,|\, p \leq p'] \geq 1 - \alpha$$

$$(3.47) \qquad Ng(p') \geq Nh(p') + np'r$$

in addition to (3.4) and (3.5).

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Solutions, φ_x, are obtained by solving the following problem for fixed n and p.

$$(3.48) \qquad \underset{\varphi}{\text{Minimize}} \sum_o^n \pi_x \varphi_x$$

subject to

$$(3.49) \qquad \sum_o^n d_x \varphi_x = h'(p) + \frac{rn}{N}$$

$$(3.50) \qquad \varphi_{k_\alpha} \geq w\left[h(p') + \frac{rk_\alpha}{N}\right]$$

$$(3.51) \qquad \sum_o^n \pi_x \varphi_x \geq h(p') + \frac{rnp'}{N}$$

and subject also to (3.4') and (3.5').

Using the data of Example 1 with $r = .3$, the following price schedule was obtained at $n^* = 39$, $p^* = .0350$.

$$\varphi_x = .5146 \qquad x = 0, 1$$
$$\varphi_x = .3869 \qquad x = 2, 3$$
$$\varphi_x = .2137 \qquad x = 4, \ldots, 7$$
$$\varphi_x = .15 \qquad x = 8, \ldots, 39$$
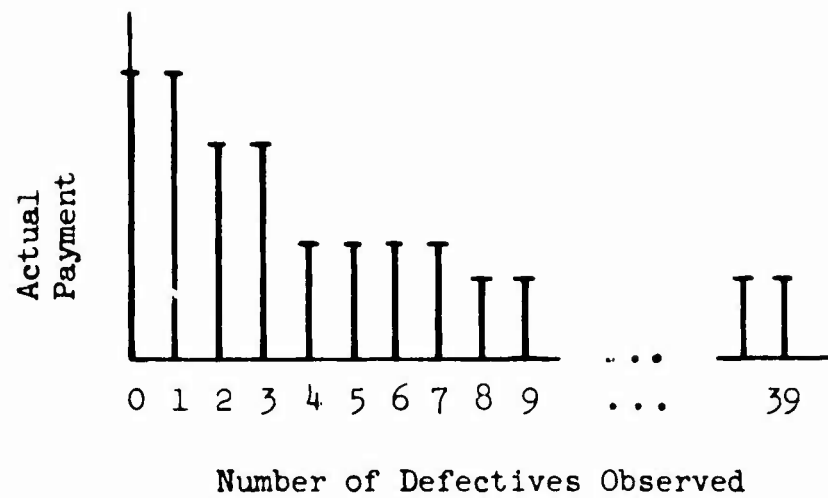
This policy is shown in Figure 16.

Figure 16

In Figure 17 and Table 4 the expected payment and profit resulting from this pricing policy are shown.
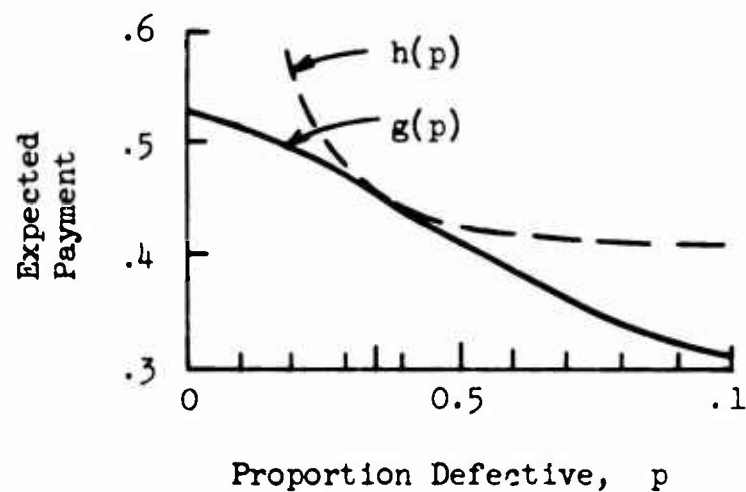
Figure 17

The consumer expected profit in this situation, .4113, is greater than the expected profit in the basic model, .3951, which certainly agrees with the procurement situation, which is more advantageous to the consumer.

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Table 4

p	Value at p	Cost at p	Expected Payment at p	Producer Profit at p	Consumer Profit at p
.01	.9271	2.4821	.5070	-1.9751	.4201
.02	.9029	.5694	.4897	- .0791	.4133
.03	.8788	.4717	.4675	- .0042	.4113
.035	.8667	.4554	.4554	.0000	.4113
.04	.8546	.4453	.4429	- .0025	.4118
.05	.8305	.4335	.4172	- .0162	.4113
.06	.8063	.4266	.3916	- .0350	.4147
.07	.7822	.4220	.3669	- .0551	.4153
.08	.7580	.4185	.3435	- .0750	.4146
.09	.7339	.4157	.3218	- .0939	.4121
.10	.7097	.4133	.3)20	- .1113	.4078

Uncertainty in Process Control

Assume that the producer does not choose p precisely, but instead chooses a mean quality, μ, at a known cost, $h(\mu)$. The distribution of the random variable p is known to be $f(p|\mu, \theta_1, \ldots, \theta_n)$ where the θ_i are known, uncontrollable parameters. We suppose that in this situation the consumer selects (n, φ) to maximize (3.52) subject to (3.54)-(3.57) knowing that the producer will select μ to maximize (3.53).

$$(3.52) \qquad T_c(\mu, \varphi, n) = NV(\mu) - Ng(\mu) - nc$$

$$(3.53) \qquad T_p(\mu, \varphi, n) = Ng(\mu) - Nh(\mu)$$

$$(3.54) \qquad pr[N\varphi(x) \geq Nw | p \leq \mu'] \geq 1 - \alpha$$

(3.55) $$\text{pr}[N\varphi(x) \leq Nv \,|\, p \geq p_b] \geq 1 - \beta$$

(3.56) $$M \geq \varphi_o \geq \cdots \geq \varphi_n \geq m$$

(3.57) $$Ng(\mu') \geq Nh(\mu')$$

For fixed n, μ, the φ_x are solutions to the linear programming problem (3.1')-(3.6') with $h(p)$, $h'(p)$ replaced by $h(\mu)$, $h'(\mu)$, and with

$$\pi_x(n,\mu) = \int_0^1 \text{pr}[x|n,p] \, f(p|\mu) \, dp \, ,$$

$$d_x(n,\mu) = \frac{\partial}{\partial \mu} \int_0^1 \text{pr}[x|n, \, p] \, f(p|\mu) \, dp \, ,$$

k_α satisfying $\sum_{j=0}^{k_\alpha} \binom{n}{j}\mu^j(1-\mu)^{n-j}$, and k_β satisfying $\sum_{j=k_\beta}^{n} \binom{n}{j} p_b^j(1-p_b)^{n-j}$.

Assume that from jobs previously attempted by the producer of the items in Example 1 we estimate that the random variable p has a beta distribution, and that while the producer can select the mean, μ, the variance is uncontrollable and is estimated at 36×10^{-6}. From this model we obtain an optimal policy at $n^* = 39$, and $\mu^* = .0339$. The policy is shown in Figure 18.

$$\varphi_x = .4692 \quad x = 0, \, 1$$
$$\varphi_x = .4608 \quad x = 2, \, 3$$
$$\varphi_x = .2138 \quad x = 4, \, \ldots \, , \, 21$$
$$\varphi_x = .1500 \quad x = 22, \, \ldots \, , \, 39$$

Figure 18

In Figure 19 and Table 5 the expected payment and profit resulting from this policy and the basic policy is shown.
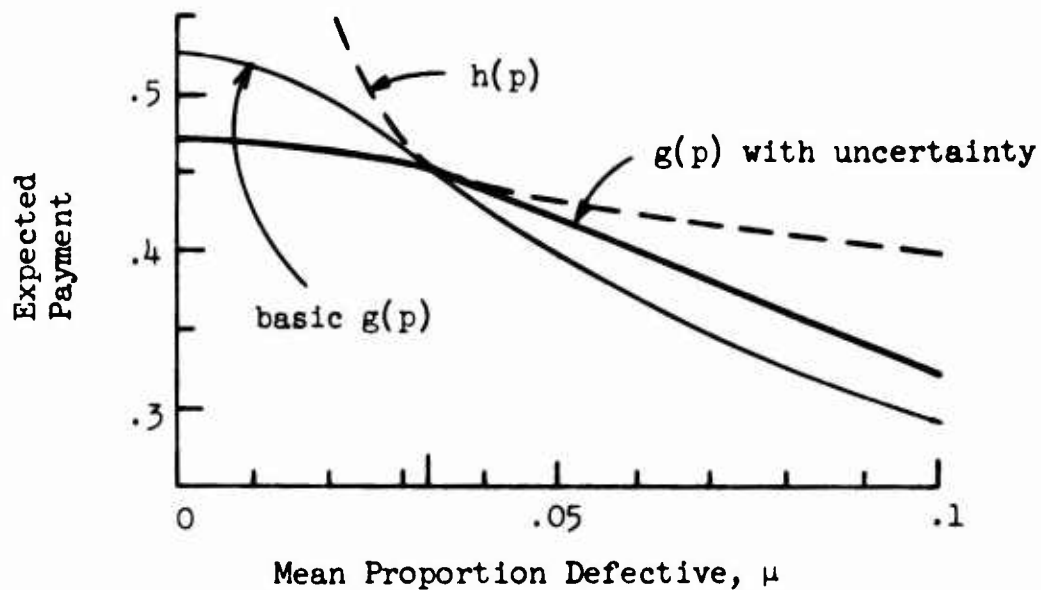
Figure 19

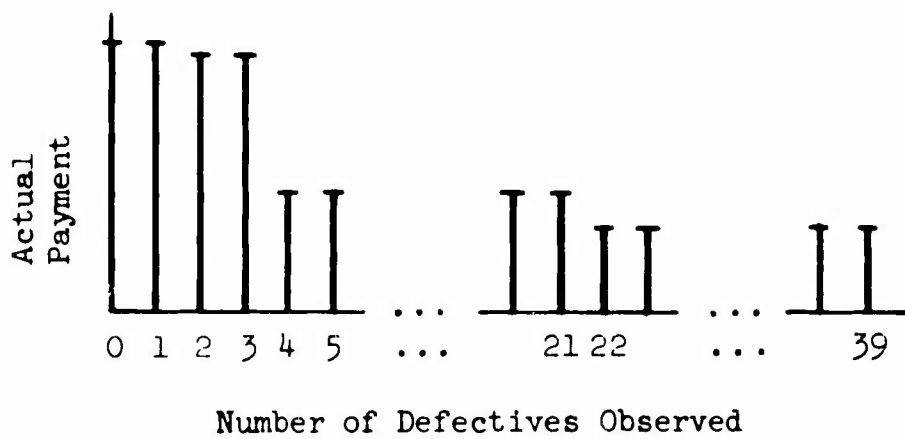A price schedule based on uncertainty in process control does not provide discrimination as fine as that in the basic case.

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Table 5

μ	Value at μ	Cost at μ	Expected Payment at μ	Producer Profit at μ	Consumer Profit at μ
.0100	.9212	2.4758	.4691	-2.0067	.4521
.0200	.8912	.5666	.4652	- .1014	.4260
.0300	.8612	.4681	.4574	- .0107	.4038
.0339	.8495	.4544	.4544	.0000	.3951
.0400	.8312	.4409	.4402	- .0007	.3910
.0500	.8012	.4281	.4229	- .0052	.3783
.0600	.7712	.4202	.4031	- .0171	.3681
.0700	.7412	.4146	.3851	- .0295	.3561
.0800	.7112	.4102	.3617	- .0485	.3435
.0900	.6812	.4064	.3409	- .0655	.3403
.1000	.6512	.4031	.3259	- .0772	.3253

Uncertainty in the Cost Function

We next assume that the cost of selecting the mean quality, μ, is not known precisely, but is $h(\mu) + \eta$, where η is a random variable with $E(\eta) = 0$, and known distribution. This situation differs from the previous case only in constraint (3.57) which we now write as

$$(3.58) \qquad pr[g(\mu') \geq h(\mu') + \eta \,|\, \mu \leq \mu'] \geq 1 - \gamma$$

which is simply $g(\mu') \geq h(\mu') + F_\eta^{-1}(1-\gamma)$, where $F_\eta^{-1}(1-\gamma)$, is the minimum u such that $F_\eta(u) \geq 1 - \gamma$, given $\mu = \mu'$.

Using the data of Example 1, with the assumption that η is normally distributed with $E(\eta) = 0$ and $\sigma_\eta = .0333\, h(\mu)$, and that p is beta distributed as in the previous example, we obtain the following policy for $\gamma = .10$. The optimal sample size is 39, and $\mu^* = .0339$. The policy is shown in Figure 20.

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$$\varphi_x = .4869 \qquad x = 0, 1$$

$$\varphi_x = .4862 \qquad x = 2, 3$$

$$\varphi_x = .2137 \qquad x = 4, \ldots , 21$$

$$\varphi_x = .1500 \qquad x = 22, \ldots , 39$$

Figure 20

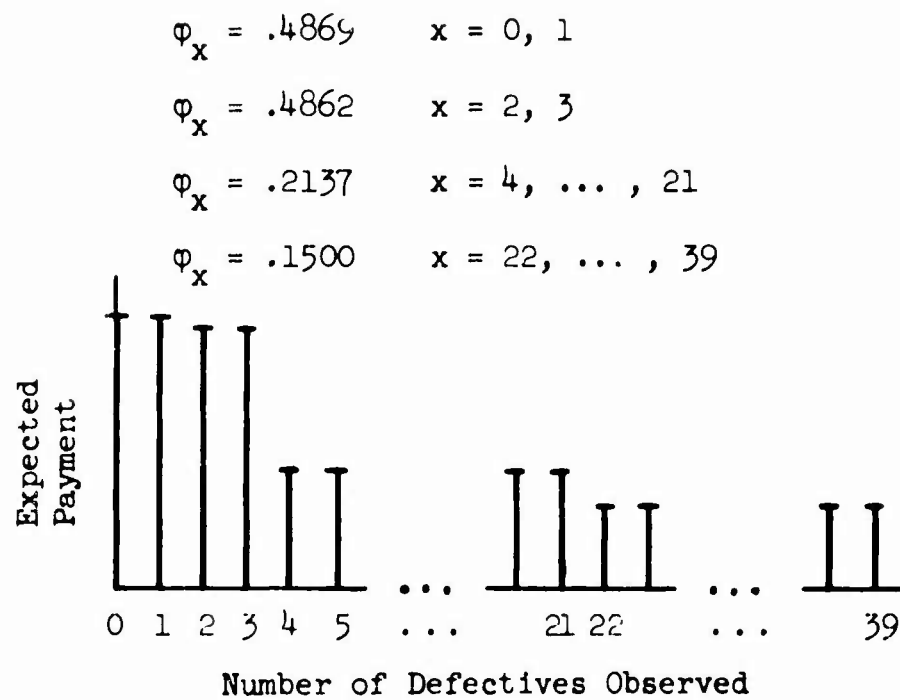In Figure 21 the expected payment resulting from this policy is shown, and Table 6 contains the expected profits.
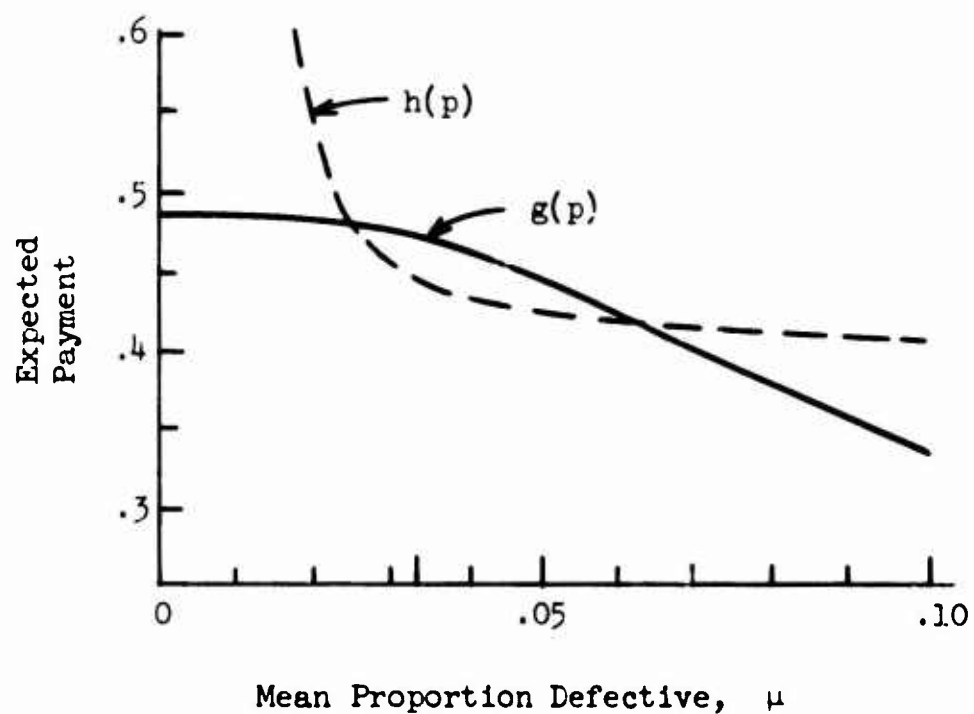
Figure 21

Table 6

μ	Value at μ	Cost at μ	Expected Payment at μ	Producer Profit at μ	Consumer Profit at μ
.01	.921	2.475	.487	-1.9880	.434
.02	.891	.566	.487	- .079	.406
.03	.861	.468	.479	.011	.382
.0339	.849	.454	.474	.020	.375
.04	.831	.440	.460	.020	.371
.05	.801	.428	.448	.020	.353
.06	.771	.420	.425	.005	.346
.07	.741	.415	.401	- .014	.340
.08	.711	.410	.381	- .029	.330
.09	.681	.406	.360	- .046	.321
.10	.651	.403	.337	- .066	.314

The producer expected profit is positive due to the constraint (3.58), as was expected since the price schedule must provide protection against cost overruns.

Applications to Lifetesting

It is apparent that incentive price schedules can be derived from the models developed here in any situation in which items can be classified as defective or non-defective. We will note one such application to the situation where the mean life of an item is of interest.

Assume that the lifetimes, t, of items produced are random variables with known distribution $f(t|\mu)$, and that the producer controls μ at a known cost $h(\mu)$. Assume also that the value to the consumer of a unit with an expected life of μ can be expressed as $V(\mu)$. Several existing reliability incentive plans place a sample of n on test,

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record the times until failure, and base the incentive payment on the average time to failure. An alternative approach is to fix the length of time, τ, for which units will be tested, consider survival to τ success, failure prior to τ, failure, and base payment on the number of failures, x. The problem is identical to the ones we have already treated with the consumer choosing n, μ, and φ_x to maximize $NV(\mu) - nc - NG(n,\mu)$, and $\int_0^\tau f(t|\mu)dt$ taking the place of p. Alternatively the sample size, n, might be predetermined and τ selected, where the unit testing cost, $c(\tau)$, depends on the test length selected.

A more interesting situation is possible in the operation called "running in." Assume, as before, that the producer controls mean life, μ, at a cost of $h(\mu)$, and that the expected value, $V(\mu)$, is known. In the "running in" operation the producer must operate items successfully for a period τ before declaring them satisfactory and delivering batches of N. The consumer can base the incentive payments on the total number, $N + x$, "run in" to achieve N satisfactory items.[13] Let the producer's expected cost be the initial production cost of the $N + x$ items, $h(\mu)[N + E(x)]$, plus the running in cost of each item, $c[N + E(x)]$. Let the consumer's problem be that of selecting τ, μ, and $\varphi(x)$ to maximize expected net gain. The only changes in the basic set of equations $(3.1)-(3.6)$, occur in (3.2) which becomes

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[13] We ignore the possibility of x becoming larger than some pre-assigned number.

$$(3.59) \qquad T_p(\mu, \varphi, n) = Ng(\mu) - \frac{N[h(\mu) + c]}{\int_\tau^\infty f(t|\mu) \, dt}, \text{[14]}$$

and in (3.6) which becomes

$$(3.60) \qquad Ng(\mu') \geq \frac{N[h(\mu') + c]}{\int_\tau^\infty f(t|\mu') \, dt}$$

where μ' maximizes (3.59) for fixed τ. For arbitrary life distributions, the prices, φ_x, can be obtained as the solution to a linear programming problem analgous to those in previous sections.

Summary

In this chapter we have developed a model of the contingent pricing situation which is more realistic than those previously reported. For fixed sample size and quality level we have derived a linear programming problem which yields as its solution an optimal pricing policy, φ_x. We have indicated conditions sufficient to insure the existence of a point (n,p) yielding such a set of prices, and described a procedure for seeking an optimal point (n^*, p^*). We modified the basic contingent pricing model by requiring that price schedules be piecewise linear, and

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[14]Since x is a random variable with a negative binomial distribution, $E(x) = \frac{Np}{q}$, and since $p = \int_0^\tau f(t|\mu) \, dt$,

$$N + E(x) = \frac{N}{q} = \frac{N}{\int_\tau^\infty f(t|\mu) \, dt}.$$

next changed the objective function to reflect a more realistic procurement situation. Finally uncertainty was permitted in both the cost of attaining an average quality and in the control of the production process, and a brief illustration was included indicating how this contingent pricing model can be applied to any item classifiable as defective or non-defective.

CHAPTER IV

CONTINGENT PRICING POLICIES FOR CONTINUOUS VARIABLES

Introduction

In Chapter III we demonstrated that the model developed for the
binomial case could be used to derive contingent pricing policies when
the characteristic of interest varied continuously. In this chapter we
develop the basic model for the continuous case, and while mathematical
programming may be used to approximate the optimal contingent pricing
policies, control theory provides interesting insights into the structure
of the optimal pricing policies.

Let μ be some characteristic of interest and let t be a con-
tinuous variable that estimates μ. Assume the consumer desires the
producer to produce at μ_o. Recall that $h(\cdot)$ is defined to include
a unit profit, and let excess profit denote the difference between $\varphi(t)$
and $h(\mu_o)$. By means of the maximum principle of L.S. Pontryagin [50]
we show that subject to weak conditions on the distribution of t, the
form of the pricing policy that both maximizes producer expected profit
at μ_o, and minimizes the mean square excess profit at μ_o is piecewise
linear in t.

The production and procurement situation we now consider differs
from that described in assumptions 1-13, Chapter III, only in the following
particulars.

1. Each item produced has a characteristic of interest, μ, which varies continuously in some interval $[\mu_1, \mu_2]$. This characteristic may be true reliability, actual propellant energy, etc.

2. Batches of N items are delivered to a consumer. From each batch n items are selected at random and tested. The result of testing the ith item is τ_i, and a statistic $t(\tau_1, \ldots, \tau_n)$ is determined by the testing, where t lies in $[t_1, t_2]$.

3. The payment schedule is $\varphi(t)$, a function of the statistic t.

4. The distribution of τ_i given μ is known and consequently $f(t|\mu)$ is known where $f(\cdot)$ denotes a probability density function. When $f(t)$ depends on parameters other than μ we assume them known and uncontrollable.

5. We assume $V(\mu)$ increasing and concave, and $h(\mu)$ increasing and strictly convex.

6. The consumer desires protection against overpayment when $\mu \leq \mu_b$, that value of μ considered poor.

In most acceptance plans and incentive arrangements, μ is estimated by t, and the problem considered is that of selecting the minimum sample size providing discrimination at an appropriate level of confidence between desirable and poor performance. Frequently test costs are high, tests are lengthy or destructive, and it is not feasible to use large sample sizes. This has led to the use of sequential sampling, accelerated life tests, and Bayesian acceptance sampling. The usual payment is either a fixed constant if the test is for the purpose of acceptance, or is generally linearly related to t in incentive contracts.

Basic Model

We assume the consumer seeks a sample size and pricing policy to maximize consumer expected net gain, knowing that given n and $\varsigma(t)$ the producer will select μ' to maximize his own expected profit. Considering assumptions 1-13, Chapter III, and assumptions 1-6, Chapter IV, we have the continuous analogue of (3.1)-(3.6) in

$$(4.1) \qquad T_c = NV(\mu) - Ng(\mu) - nc \; ,$$

$$(4.2) \qquad T_p = Ng(\mu) - Nh(\mu) \; ,$$

$$(4.3) \qquad pr[N\varphi(t) \geq w | \mu \geq \mu'] \geq 1 - \alpha \; ,$$

$$(4.4) \qquad pr[N\varphi(t) \leq v | \mu \leq \mu_b] \geq 1 - \beta \; ,$$

$$\varphi(t_1) \geq m \; ,$$

$$(4.5) \qquad \varphi(t_2) \leq M \; ,$$

$$\frac{d\varphi(t)}{dt} \equiv \dot{\varphi}(t) \geq 0 \; ,$$

$$(4.6) \qquad g(\mu') \geq h(\mu') \; .$$

Proposition 1

Let $r(t|\mu,n) = \frac{\partial}{\partial \mu} f(t|\mu,n)$. If for fixed n, μ, say (n,μ_o) there exists $\varphi(t)$ maximizing (4.1) and (4.2) at μ_o and for which (4.3)-(4.6) hold, then $\varphi(t)$ minimizes (4.1') and (4.2')-(4.6') hold.

$$(4.1') \qquad g(\mu_o) = \int_{t_1}^{t_2} \varphi(t) f(t|\mu_o,n) \, dt$$

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$$(4.2') \qquad \int_{t_1}^{t_2} \varphi(t) \; r(t|\mu_o,n) \; dt = h'(\mu_o)$$

$$(4.3') \qquad \varphi(t_\alpha) \geq w \; , \quad \int_{t_\alpha}^{t_2} f(t|\mu_o,n) \; dt \geq 1 - \alpha$$

$$(4.4') \qquad \varphi(t_\beta) \leq v \; , \quad \int_{t_1}^{t_\beta} f(t|\mu_b,n) \; dt \geq 1 - \beta$$

$$\varphi(t_1) \leq m$$

$$(4.5') \qquad\qquad \varphi(t_2) \geq M$$

$$\dot{\varphi}(t) \geq 0$$

$$(4.6') \qquad \int_{t_1}^{t_2} \varphi(t) \; f(t|\mu_o,n) \; dt \geq h(\mu_o)$$

The proof is identical to that of Theorem 1, Chapter III, and is omitted.

Note that approximations to $\varphi(t)$ are readily obtained by partitioning $[t_1, t_2]$ into a finite number of intervals, and using (3.10) to derive an equivalent linear programming problem containing only five constraints. The approximation can be made extremely precise with little loss in computational efficiency since the computing time required to derive solutions to linear programming problems increases much more rapidly with the number of constraints than the number of variables. Due to the number of constraints the discrete approximation obtained

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through linear programming need have at most six distinct price levels. In the next section we will show that the maximum principle also yields the policy, $\varphi(t)$, but linear programming procedures are more readily available in industrial and government computing centers than are the numerical analysis routines required to determine the exact $\varphi(t)$ by the maximum principle. Consequently from the point of view of implementing contingent pricing policies, the use of mathematical programming is preferable.

Maximum Principle

Let x, p, denote n-dimensional vectors, and x_i, p_i their components. Let λ_i be unknown constants, b_i, x_i^1 known constants, $\underline{\Phi}(t)$, $\overline{\Phi}(t)$ known functions of t, and R, q, known functions of their arguments, continuously differentiable with respect to each variable. We seek $\varphi(t)$, $\underline{\Phi}(t) \leq \varphi(t) \leq \overline{\Phi}(t)$, which minimizes (4.7) and satisfies (4.10). We will call such a minimizing $\varphi(t)$ optimal.

$$(4.7) \qquad \rho = \sum_{i=1}^{n} b_i x_i(t_2)$$

$$(4.8) \qquad \frac{dx_i(t)}{dt} \equiv \dot{x}_i(t) = q_i(x, \varphi, t) \ , \quad i = 1, \ldots, n$$

$$(4.9) \qquad x_i(t_1) = x_i^1 \ , \quad i = 1, \ldots, n$$

$$(4.10) \qquad R(x_1(t_2), \ldots, x_n(t_2)) = 0$$

$$(4.11) \qquad \dot{p}_i(t) = - \sum_{j=1}^{n} p_j \frac{\partial q_j}{\partial x_i} \ , \quad i = 1, \ldots, n$$

$$(4.12) \qquad p_i(t_2) = -\left[b_i + \lambda_i \frac{\partial R}{\partial x_i(t_2)}\right], \quad i = 1, \ldots, n$$

$$(4.13) \qquad H(x,p,\varphi,t) = \sum_{i=1}^{n} p_i q_i$$

The maximum principle states that in order for $\varphi(t)$ to be optimal it is necessary that there exist $p_1(t), \ldots, p_n(t)$ not all zero, such that $H(x,p,\varphi,t)$ is maximized at $\varphi(t)$ for every t, $t_1 \leq t \leq t_2$. When every $\dot{x}_i(t)$ is linear in $x(t)$ and contains only an additive function of $\varphi(t)$ as in (4.14) the stated conditions are also sufficient.

$$(4.14) \qquad \dot{x}_i(t) = \sum_{k=1}^{n} a_{ik} x_k(t) + \omega[\varphi(t)] .$$

Application of the Maximum Principle

The maximum principle is not directly applicable to (4.1')-(4.6') due to the constraint $\dot{\varphi}(t) \geq 0$, but by a transformation similar to (3.10) we seek $y(t)$, the derivative of $\varphi(t)$. Define $y(t)$ and $\varphi(t)$ by (4.15)-(4.17) where the integrals are Stieltjes integrals.

$$(4.15) \qquad \int_t^{t_2} y(t) \, dt = \int_t^{t_2} d\varphi(t)$$

$$(4.16) \qquad \varphi(t_2) = M - y(t_2)$$

$$(4.17) \qquad \varphi(t) = \varphi(t_2) - \int_t^{t_2} y(t) \, dt$$

Define $F(t)$, $D(t)$, $x_i(t)$, $i = 1, \ldots, 5$, by

$$(4.18) \qquad F(t) = \int_{t_1}^{t} f(t|\mu_0,n) \, dt$$

$$(4.19) \qquad D(t) = \int_{t_1}^{t} r(t|\mu_0,n) \, dt$$

$$(4.20) \qquad x_1(t) = -\int_{t_1}^{t} y(t) \, F(t) \, dt$$

$$(4.21) \qquad x_2(t) = -\int_{t_1}^{t} y(t) \, D(t) \, dt$$

$$(4.22) \quad x_3(t) = -\int_{t_1}^{t} y(t) \, J_1(t) \, dt \,, \quad \text{where} \quad J_1(t) = 0, \; t_1 \le t < t_\alpha \,,$$
$$= 1, \; t_\alpha \le t \le t_2 \,,$$

$$(4.23) \quad x_4(t) = -\int_{t_1}^{t} y(t) \, J_2(t) \, dt \,, \quad \text{where} \quad J_2(t) = 0, \; t_1 \le t < t_\beta \,,$$
$$= 1, \; t_\beta \le t \le t_2 \,,$$

$$(4.24) \quad x_5(t) = -\int_{t_1}^{t} y(t) \, J_3(t) \, dt \,, \quad \text{where} \quad J_3(t) = 1, \; t_1 \le t \le t_2 \,.$$

We seek $y(t) \ge 0$ which maximizes $x_1(t_2)$ and satisfies

$$(4.25) \qquad x_1(t_2) \ge h(\mu_0) \,,$$

$$(4.26) \qquad x_2(t_2) = h'(\mu_0) - \varphi(t_2) \ ,$$

$$(4.27) \qquad x_3(t_2) \geq w - \varphi(t_2) \ ,$$

$$(4.28) \qquad x_4(t_2) \leq v - \varphi(t_2) \ ,$$

$$(4.29) \qquad x_5(t_2) \geq m - \varphi(t_2) \ .$$

From (4.7), $b_1 = 1$, b_2, ... , $b_5 = 0$. The $\dot{x}_i(t)$ defined by (4.20)-(4.24) are not functions of $x(t)$. Therefore from (4.11) $\dot{p}_i(t) = 0$, $i = 1$, ... , 5, and from (4.12) $p_i(t) \equiv p_i(t_2)$. Thus $p_1(t_2) = -[1 + \lambda_1]$, $p_i(t_2) = -\lambda_i$, $i = 2$, ... , 5. Denoting $p_i(t)$ by constants ψ_i, we obtain H from (4.13), and (4.20)-(4.24).

$$(4.30) \qquad H = y[\psi_1 F + \psi_2 D + \psi_3 J_1 + \psi_4 J_2 + \psi_5 J_3]$$

The maximum principle states that H as a function of y must be maximized at every t, $t_1 \leq t \leq t_2$.[1] Abbreviate H by $H = yK(t)$. For $K(t) < 0$, $y(t) = 0$, and for $K(t) \geq 0$, $y(t) = +\infty$. If an optimal $\varphi(t)$ exists there must be ψ_1, ... , ψ_5 such that $K(t)$ is never positive and is zero at only a finite number of points, t_j. At these points $\int_{t_j^-}^{t_j^+} y(t) \, dt = \varphi(t_j^+) - \varphi(t_j^-)$, the jump in $\varphi(t)$ at t_j.

The exact number of such impulse points depends on $f(t)$. For unimodal $f(t)$, $-D(t)$ is unimodal, and since $F(t)$ increases in t,

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[1] For the remainder of the chapter t will be understood to be in $[t_1, t_2]$ without additional specification.

there can be at most five points of tangency with $K(t) = 0$. If there are exactly five, one must occur at $t = t_1$ due to $J_3(t)$, one at $t = t_\alpha$ due to $J_1(t)$, and one at $t = t_\beta$ due to $J_2(t)$. The location of the remaining two depends on $f(t)$. Thus the maximum principle indicates that $\varphi(t)$ is piecewise constant with no more than six distinct price levels, and yields additional information as to where the price levels may change.

Discontinuous price schedules are unattractive to a producer, which partially accounts for the use of piecewise linear policies in practice. With the problem formulated as in (4.15)-(4.29) we need only add the constraint $y(t) \leq Y$ where Y is positive, to replace the jumps in $\varphi(t)$ by linear subarcs. This also reduces the set of points (n,μ) on which solutions will exist. This may be thought of as "forcing" the selection of a piecewise linear policy.

From the following theorem we can infer that by seeking agreement to a piecewise linear pricing policy, the consumer is seeking that policy which both minimizes the producer's mean square excess profit at μ_o, and maximizes producer expected profit at μ_o. We define the following problem which is equivalent to the original contingent pricing problem (4.1)-(4.6) except that for clarity we delete the constraints (4.3), (4.4), and the monotonicity requirement, $\dot\varphi(t) \geq 0$.

(4.31)
$$x_o(t) = \int_{t_1}^{t} [\varphi(t) - h(\mu_o)]^2 \, f(t|\mu_o,n) \, dt$$

(4.32)
$$x_1(t) = \int_{t_1}^{t} \varphi(t) \, r(t|\mu_o,n) \, dt$$

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$$(4.33) \qquad x_2(t) = \int_{t_1}^{t} \varphi(t) \, f(t|\mu_0,n) \, dt$$

$$(4.34) \qquad m \le \varphi(t) \le M$$

$$(4.35) \qquad x_1(t_2) = h'(\mu_0)$$

$$(4.36) \qquad x_2(t_2) \ge h(\mu_0)$$

We seek that $\varphi(t)$ satisfying (4.34) which minimizes $x_0(t_2)$ and yields $x_1(t_2)$, $x_2(t_2)$ satisfying (4.35), (4.36). We call such a $\varphi(t)$ optimal for (4.31)-(4.36).

Theorem 1

If $\varphi(t)$ is optimal for (4.31)-(4.36) and if $\frac{\partial}{\partial \mu} \ln f(t|\mu_0) = A_1(\mu_0) + A_2(\mu_0) \, t$ then it is necessary and sufficient that $\varphi(t)$ be a piecewise linear policy of the form (4.37).

$$(4.37) \qquad \begin{aligned} \varphi(t) &= m & t_1 &\le t < t_m \\ \varphi(t) &= s(t) & t_m &\le t < t_M \\ \varphi(t) &= M & t_M &\le t \le t_2 \end{aligned}$$

Proof:

By (4.7)-(4.13) $H(x,p,\varphi,t)$ for this problem is

$$(4.38) \qquad H = - [\varphi(t) - h(\mu_0)]^2 \, f(t) + \psi_1 \varphi(t) \, r(t) + \psi_2 \varphi(t) \, f(t) \, .$$

The ψ_i are undetermined constants. The maximum principle states that if $\varphi(t)$ is optimal for (4.31)-(4.36) it is necessary and sufficient

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that H as a function of φ be maximized at every t. If $\varphi(t)$ is
not m or M, a necessary and sufficient condition for H to be
maximized by φ is

(4.39)
$$\varphi(t) = \left(\frac{\psi_1}{2}\right) \frac{r(t)}{f(t)} + \frac{\psi_2}{2} + h(\mu_o)$$

since it is easily verified that $\dfrac{\partial^2 H}{\partial \varphi^2} < 0.$

By definition $r(t) = \dfrac{\partial}{\partial \mu} f(t|\mu)$, and $\dfrac{r(t)}{f(t)} = \dfrac{\partial}{\partial \mu} \ln f(t|\mu)$ which
is linear in t by hypothesis. Denote the right hand side of (4.39)
by $s(t)$.

If $s(t) < m$ or $s(t) > M$ for every t, (4.35) cannot be satisfied
for $h'(\mu_o) > 0$. Therefore if an optimal policy exists there is at least
one t for which $m < s(t) < M$. Let t_m satisfy $s(t_m) = m$ and let
t_M satisfy $s(t_M) = M$. (4.37) follows.

Remarks on Theorem 1

The statement $\dfrac{\partial}{\partial \mu} \ln f(t|\mu) = A_1(\mu) + A_2(\mu) t$ implies that $f(t|\mu)$
is a member of the exponential family of distributions,
$f(t|\mu) = J(\mu) \exp [Q(\mu) U(t)] h(t)$, but we require additionally that
$U(t)$ be linear in t. The normal, gamma, binomial, poisson, and geo-
metric distributions satisfy this linearity requirement.

By including the constraint $g(\mu_o) \geq h(\mu_o)$ we introduced the
constant term $\dfrac{\psi_2}{2}$ into $\varphi(t)$ but this does not affect the form of the
price schedule. There is also no need to reintroduce the constraint
$\dot{\varphi}(t) \geq 0$ since by Theorem 1 if an optimal $\varphi(t)$ exists and
$h'(\mu_o) > 0$, $\dot{s}(t)$ must be positive.

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The effect of introducing (4.3), (4.4) is to change the constraint set for $\varphi(t)$. Jump discontinuities may occur at $t = t_\alpha$ and $t = t_\beta$. Since jump discontinuities are not seen in practice we infer that either the consumer and producer do not explicitly specify prices w and v, or that n and (M - m) are sufficiently large so that (4.3') and (4.4') are not active constraints.

There is an interesting connection with estimation which we cannot pursue here.[2] If f(t) is a member of the exponential family and U(t) is linear in t, it can be shown that t is a sufficient statistic for the τ_i, and that t is the minimum variance estimator of μ among all estimates with the same bias. Therefore if we define the quadratic loss function (4.40), and the risk function (4.41),

$$(4.40) \qquad\qquad L = (\mu - t)^2$$

$$(4.41) \qquad\qquad R(\mu, t) = K(\mu)\ E(\mu - t)^2$$

t is the estimate which minimizes risk.

Existence of Piecewise Constant Pricing Policies

In order for a price schedule satisfying (4.2')-(4.6') to exist at some (n, μ_o) the sample size must be sufficiently large to permit discrimination between μ_o and μ_b, and the price levels M, m, w, and v must be related to $h(\mu_o)$ in a manner that permits $\varphi(t)$ to satisfy (4.2')-(4.6'). Proposition 2 states these conditions. The proof is similar to that of Proposition 1, Chapter III and is sketched briefly.

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[2] See Lindgren, [51, p. 220].

Proposition 2

Let n and μ_o be fixed and assume $t_\beta < t_\alpha < \mu_o$ where t_α, t_β are given by $(4.3')$, $(4.4')$. Let $D_a^b = \int_a^b r(t|\mu,n)\,dt$ and $P_a^b = \int_a^b f(t|\mu,n)\,dt$. Then there exists a $\varphi(t)$ satisfying $(4.2')$-$(4.6')$ at (n,μ_o) if

$$(4.42) \qquad (w-v)\, D_{t_\beta}^{t_2} \le h'(\mu_o) \le (m-M)\, D_{t_1}^{t_\alpha} + (w-M)\, D_{t_\alpha}^{\mu_o}$$

and

$$(4.43) \quad h(\mu_o) \le \lambda\left[(m-M)\, P_{t_1}^{t_\alpha} + (w-M)\, P_{t_\alpha}^{\mu_o} + M\right] + (1-\lambda)\left[(v-w)\, P_{t_1}^{t_\beta} + w\right]$$

hold where λ is given by (4.46).

Proof:

Let $\overline{g}'(\mu_o)$, $\underline{g}'(\mu_o)$ denote the maximum and minimum values of $g'(\mu_o)$ for any policy, $\varphi(t)$, satisfying $(4.3')$-$(4.5')$, and let $\overline{g}(\mu_o)$, $\underline{g}(\mu_o)$ denote the expected payment corresponding to these policies.

Since $t_\alpha < \mu_o$, $g'(\mu_o)$ is at most $mD_{t_1}^{t_\alpha} + wD_{t_\alpha}^{\mu_o} + MD_{\mu_o}^{t_2}$ or,

$$(4.44) \qquad \overline{g}'(\mu_o) = (m-M)\, D_{t_1}^{t_\alpha} + (w-M)\, D_{t_\alpha}^{\mu_o}.$$

By (4.42) this is not less than $h'(\mu_o)$. Similarly

$$(4.45) \qquad \underline{g}'(\mu_o) = (w-v)\, D_{t_\beta}^{t_2},$$

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and by (4.42) this is not greater than $h'(\mu_o)$. Therefore for some λ, $0 \leq \lambda \leq 1$, $h'(\mu_o) = \lambda \bar{g}'(\mu_o) + (1-\lambda) \underline{g}'(\mu_o)$ and thus (4.2') is satisfied. From this we obtain

$$(4.46) \qquad \lambda = \frac{h'(\mu_o) - \underline{g}'(\mu_o)}{\bar{g}'(\mu_o) - \underline{g}'(\mu_o)} \quad .$$

By (4.43) $h(\mu_o) \leq \lambda \left[(m-M) \; P_{t_1}^{t_\alpha} + (w-M) \; P_{t_\alpha}^{\mu_o} + M \right] + (1-\lambda) \left[(v-w) \; P_{t_1}^{t_\beta} + w \right]$.

After rearrangement

$$(4.47) \qquad h(\mu_o) \leq \lambda \bar{g}(\mu_o) + (1-\lambda) \; \underline{g}(\mu_o)$$

which satisfies (4.6'), and completes the proof.

With the specification that $t = \dfrac{\sum_1^n \tau_i}{n}$ and the assumption that t is normally distributed, we state conditions sufficient for the convexity of R, the set of points (n,μ) feasible for (4.2')-(4.6'). The importance of having a convex constraint region is that we can be certain of determining (n_o, μ_o) maximizing $\psi(n,\mu) = NV(\mu) - Nh(\mu) - nc$.

Proposition 3

Let $t = \dfrac{1}{n} \sum_1^n \tau_i$ and assume that t is normally distributed with mean μ and known variance $\dfrac{1}{n} \sigma_\tau^2$. Let $t_\alpha < \mu_o$, $t_\beta > \mu_b$ and let $\underline{n}(\mu_o)$ be the minimum n such that $t_\beta < t_\alpha$ for $\mu = \mu_o$. Then $t_\beta < t_\alpha$ for every $n \geq \underline{n}(\mu_o)$.

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Proof:

It suffices to show that t_β decreases and t_α increases as n increases. Let $f_{N*}(\cdot)$ denote the standardized normal density function and $F_{N*}(\cdot)$ denote the cumulative function. Let u_β satisfy $F_{N*}(u_\beta) = 1 - \beta$ and u_α satisfy $F_{N*}(u_\alpha) = \alpha$.

Since $t_\beta = \dfrac{u_\beta \sigma_\tau}{\sqrt{n}} + \mu_b$ and $u_\beta > 0$ by hypothesis, t_β decreases in n. Similarly $t_\alpha = \dfrac{u_\alpha \sigma_\tau}{\sqrt{n}} + \mu_o$ and since $u_\alpha < 0$ by hypothesis t_α increases in n.

Proposition 4 states conditions under which R is convex in μ for fixed n_o, and in n for fixed μ_o. The proof is identical to that of Proposition 2, Chapter III, and is omitted.

Proposition 4

Let $\underline{\mu}(n_o)$ be the minimum μ for which both

$$\int_{t_\alpha}^{t_2} f(t \mid \underline{\mu}(n_o), n_o)\, dt \geq 1 - \alpha \quad \text{and} \quad t_\alpha > t_\beta. \quad \text{Let } \bar{\mu} \text{ be chosen}$$

arbitrarily in $[t_1, t_2]$ so that $h'(\bar{\mu}) < \infty$ and $\underline{\mu}(n_o) < \bar{\mu}$. Let $\bar{g}'(\mu)$ be minimized at μ' and $\underline{g}'(\mu)$ be maximized at μ'', where μ', $\mu'' \in [\underline{\mu}(n_o), \bar{\mu}]$. Then $(4.2')$-$(4.6')$ can be satisfied at every (n_o, μ), $\underline{\mu}(n_o) \leq \mu \leq \bar{\mu}$, if

$$(4.48) \qquad (m-M)\, D_{t_1}^{t_\alpha}(\mu') + (w-M)\, D_{t_\alpha}^{\bar{\mu}}(\mu') \geq h'(\bar{\mu}) ,$$

$$(4.49) \qquad (w-v)\, D_{t_\beta}^{t_2}(\mu'') \leq h'(\underline{\mu}(n_o)) ,$$

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(4.50) $$h(\mu_o) \stackrel{=}{\_} \lambda \bar{g}(\mu') + (1-\lambda) \underline{g}(\mu") ,$$

hold where

(4.51) $$\lambda = \frac{h'(\mu) - \underline{g}'(\mu")}{\bar{g}'(\mu') - \underline{g}'(\mu")} .$$

For $n_2 > n_o$ and fixed μ_o let $\bar{g}'(\mu_o)$ be minimized at n', $n_o \leq n' \leq n_2$ and $\underline{g}'(\mu)$ be maximized at $n"$, $n_o \leq n" \leq n_2$. If (4.48)-(4.50) hold after replacing μ' by n' and $\mu"$ by $n"$ then R is convex in n, $n_o \leq n \leq n_2$ for fixed μ_o.

While under these conditions we can compute (n_o, μ_o) maximizing $\psi(n,\mu)$, there is still little that can be said about the concavity of $G(n,\mu)$. However the generalized Fibonacci search procedure described in Chapter III will yield better results for the conditions existing in Chapter IV. This is due to the fact that R is convex over a larger region than in the cases described in Chapter III, allowing us to search $G(n,\mu)$ without introducing artificial variables, as in (3.1^o).

Payment Functions of Several Variables

In many cases incentives apply to several characteristics simultaneously. An example of this is an incentive on production schedule, equipment reliability, and maintainability. Let μ and t denote vectors, μ^i, t^i components, and μ^i_v fixed values. We assume the desired values of k performance or quality characteristics, $\mu^i_o, \ldots, \mu^k_o$ are known, that μ^i is estimated by $t^i = \frac{1}{n} \sum_{j=1}^{n} \tau^i_j$, and that $f(t|\mu)$ is known.

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A procedure frequently observed is the assignment of constant weights, $b_i \geq 0$, $\sum_{i=1}^{k} b_i = 1$, to the performance characteristics, and definition of the random variable, $Z = \sum_{i=1}^{k} b_i t^i$. A single policy $\varphi(Z)$ is then formulated. Assume the distribution of t to be multivariate normal with known parameters. Then Z is normally distributed with known mean and variance and the situation is similar to the single variable cases discussed in an earlier section of Chapter IV. For example we seek $\varphi(Z)$ to minimize

$$(4.52) \qquad \int_{z_1}^{z_2} [\varphi(Z) - h(\mu_0)]^2 \, f(Z|\mu_0) \, dZ \, ,$$

subject to

$$(4.53) \qquad \int_{z_1}^{z_2} \varphi(Z) \, \frac{\partial f(Z|\mu_0)}{\partial \mu^i} \, dZ = \frac{\partial h(\mu_0)}{\partial \mu^i} \qquad i = 1, \ldots, k \, ,$$

$$(4.54) \qquad \int_{z_1}^{z_2} \varphi(Z) \, f(Z|\mu_0) \, dZ \geq h(\mu_0) \, ,$$

$$(4.55) \qquad\qquad m \leq \varphi(Z) \leq M \, .$$

Let $s(Z) = h(\mu_0) + \frac{\Psi_{k+1}}{2} + \sum_{i=1}^{k} \frac{\Psi_i}{2} \frac{\partial}{\partial \mu^i} \ln f(Z|\mu_0)$. By Theorem 1, Chapter IV, the optimal policy is given by

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$$\begin{array}{lll}
& \varphi(Z) = m & s(Z) < m , \\
(4.56) & \varphi(Z) = s(Z) & m \le s(Z) \le M , \\
& \varphi(Z) = M & M < s(Z) .
\end{array}$$

Since Z is normally distributed $s(Z)$ is linear in Z. Constraints equivalent to (4.3), (4.4) can be introduced with no essential changes in the formulation, and as an alternative to (4.52) we could introduce a linear objective function to allow computation by means of the linear programming methods of Chapter III. The key simplification is the combination of $t^1, \ldots , t^k$ into the single variable Z, by consumer specification of the weights, b_i.

It is possible to also include production schedule in $\varphi(Z)$ by defining a variable μ^o related to schedule in such a way that increases in μ^o are desirable. We then assume that if the producer aims at μ_o^o he attains t^o where $t^o, \ldots , t^k$ have a multivariate normal distribution with $E(t) = \mu$ and known covariance matrix. Thus the situation is identical to that ju described. However, pricing functions are usually formulated separately for performance variables and production schedule. That is $\varphi(t^o,Z) = \varphi_1(t^o) + \varphi_2(Z)$. Since

$$\int_{t^o}\int_Z \varphi(t^o,Z)\ f(t^o,Z|\mu^o, \ldots , \mu^k)\ dt^o dZ = \int_{t^o} \varphi_1(t^o)\ f_1(t^o)\ dt^o$$

$$+ \int_Z \varphi_2(Z)\ f_2(Z)\ dZ$$ we now seek two pricing policies, $\varphi_1(t^o)$ and $\varphi_2(Z)$ to minimize

$$(4.57) \quad \int_{t_1}^{t_2} \varphi_1(t^o)\ f_1(t^o|\mu_o)\ dt + \int_{z_1}^{z_2} \varphi_2(Z)\ f_2(Z|\mu_o)\ dZ ,$$

subject to

(4.58)
$$\frac{\partial}{\partial \mu^i} [g_Z(\mu_o) - h(\mu_o)] = 0 , \qquad i = 1, \ldots, k ,$$

(4.59)
$$\frac{\partial}{\partial \mu^o} [g_{t^o}(\mu_o) - h(\mu_o)] = 0 ,$$

(4.60)
$$g_Z(\mu_o) + g_{t^o}(\mu_o) \geq h(\mu_o) ,$$

(4.61)
$$m_Z \leq \varphi_2(Z) \leq M_Z ,$$

(4.62)
$$m_{t^o} \leq \varphi_1(t^o) \leq M_{t^o} .$$

Note that in this formulation we are not required to choose the weights to be applied to overall performance, Z, and schedule, t^o. Arbitrarily precise approximations to $\varphi_1(\cdot)$ and $\varphi_2(\cdot)$ can be obtained by the methods of Chapter III.

In many situations payment based on a constant weighting of the μ^1 does not reflect the consumer desires outside a small region about μ_o, and there may be considerable uncertainty concerning the correct weights within that region. We require a price schedule which is a joint function of the variables $t^1, \ldots, t^k$. Specifically, consider the determination of a price function $\varphi(t^1, t^2)$ where t^1, t^2 estimate different performance or quality characteristics, and the joint distribution of t^1 and t^2 is known. We seek $\varphi(t^1, t^2)$ minimizing

(4.63)
$$\int_{t^1} \int_{t^2} \varphi(t^1, t^2) \, f(t^1, t^2 | \mu_o^1, \mu_o^2) \, dt^1 dt^2 ,$$

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subject to

(4.64) $$\frac{\partial g(\mu_o^1, \mu_o^2)}{\partial \mu^i} = \frac{\partial h(\mu_o^1, \mu_o^2)}{\partial \mu^i} \; , \qquad i = 1, 2,$$

(4.65) $$m \leq \varphi(t^1, t^2) \leq M \; ,$$

(4.66) $$g(\mu_o^1, \mu_o^2) \geq h(\mu_o^1, \mu_o^2) \; ,$$

(4.67) $$\frac{\partial \varphi(t^1, t^2)}{\partial t^i} \geq 0 \; , \qquad i = 1, 2.$$

By partitioning the interval $[t_1^1, t_2^1]$ into ν_1 subintervals and the interval $[t_1^2, t_2^2]$ into ν_2 subintervals, we have the following approximating linear programming problem where $t^1 = t_1^1 + i(t_2^1 - t_1^1)/\nu_1$, $t^2 = t_1^2 + j(t_2^2 - t_1^2)/\nu_2$. Minimize

(4.68) $$\sum_{i=0}^{\nu_1} \sum_{j=0}^{\nu_2} \varphi_{ij} f_{ij}(\mu_o^1, \mu_o^2) \; ,$$

subject to

(4.69) $$\sum_{i=0}^{\nu_1} \sum_{j=0}^{\nu_2} \varphi_{ij} \frac{\partial f_{ij}}{\partial \mu^i} = \frac{\partial h(\mu_o^1, \mu_o^2)}{\partial \mu^i} \; , \qquad i = 1, 2,$$

(4.70) $$g(\mu_o^1, \mu_o^2) \geq h(\mu_o^1, \mu_o^2) \; ,$$

(4.71) $$m \leq \varphi_{0j} \leq \varphi_{1j} \leq \cdots \leq \varphi_{\nu_1 j} \leq M \; , \quad j = 0, \ldots, \nu_2 \; ,$$

(4.72) $$m \leq \varphi_{10} \leq \varphi_{11} \leq \cdots \leq \varphi_{1\nu_2} \leq M \ , \quad i = 0, \ldots, \nu_1 \ .$$

This problem can be solved by standard linear programming methods, but the computation required has increased considerably. Even a relatively coarse partition of $[t_1^1, t_2^1]$ and $[t_1^2, t_2^2]$ into 20 subintervals results in a problem containing 43 constraints and 441 variables, indicating extension to more than two variables is impractical. However the important two variable problem is that of determining an optimal joint price function of schedule performance, t^o, and the overall measure of performance and quality, Z. This joint price function can be derived by the method just outlined.

A realistic and tractable alternative to seeking an optimal $\varphi(t^1, \ldots, t^k)$ from the class of all bounded, monotonic, functions is to seek that $\varphi(t^1, \ldots, t^k)$ yielding an expected payment satisfying specified constraints. One natural and desirable expected payment is

(4.73) $$g(\mu) = V(\mu) - [V(\mu_o) - h(\mu_o)] \ .$$

If the actual outcome $(t^1, \ldots, t^k)$ is expected to be close to μ_o, or if $V(\mu)$ is not changing rapidly, $g(\mu)$ can be approximated adequately by the first three terms of a Taylor series as

(4.74) $$g(\mu) \approx h(\mu_o) + \sum_1^k \frac{\partial V(\mu_o)}{\partial \mu^i} (\mu^i - \mu_o^i)$$

$$+ \frac{1}{2} \sum_{i,j=1}^k (\mu^i - \mu_o^i) \frac{\partial^2 V(\mu_o)}{\partial \mu^i \partial \mu^i} (\mu^j - \mu_o^j) \ .$$

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Under the assumption that the distribution of t is multivariate normal, and that the covariance matrix $G = [g_{ij}]$ is known, $\varphi(t)$ need only be a quadratic function of the variables t^i to yield $E[\varphi(t)] = g(\mu)$. For the $g(\mu)$ above,

$$(4.75) \qquad \varphi(t) = a + \sum_{i=1}^{k} b_i t^i + \sum_{i,j=1}^{k} c_{ij} t^i t^j ,$$

where

$$(4.76) \qquad a = h(\mu_o) - \sum_{i=1}^{k} \frac{\partial V(\mu_o)}{\partial \mu^i} \mu_o^i + \frac{1}{2} \sum_{i,j=1}^{k} (\mu_o^i \mu_o^j - g_{ij}) \frac{\partial^2 V(\mu_o)}{\partial \mu^i \partial \mu^i} ,$$

$$(4.77) \qquad b_i = \frac{\partial V(\mu_o)}{\partial \mu^i} - \sum_{j=1}^{k} \mu_o^j \frac{\partial^2 V(\mu_o)}{\partial \mu^i \partial \mu^j} ,$$

and

$$(4.78) \qquad c_{ij} = \frac{1}{2} \frac{\partial^2 V(\mu_o)}{\partial \mu^i \partial \mu^j} .$$

Piecewise Linear Price Schedules

We refer to a piecewise linear pricing policy of the form (4.79) as a "linear" policy, and to a piecewise constant pricing policy satisfying (4.2')-(4.6') as a "basic" policy.

$$(4.79) \qquad
\begin{aligned}
\varphi(t) &= m_o \geq m & t &\leq a \\
\varphi(t) &= m_o + (t-a) s & a &\leq t \leq b \\
\varphi(t) &= M_o \leq M & t &\geq l \\
s &= \frac{M_o - m_o}{b - a}
\end{aligned}$$

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In Chapter III we described a procedure for deriving linear policies, and in Chapter IV indicated the criterion that leads naturally to such policies. The attractive features of linear policies are continuity and ease of interpretation, however linear policies can never be more profitable for the consumer than basic policies, and in this section we investigate the extra cost incurred by the consumer resulting from the use of an optimal linear policy.

Let $\varphi(t)$ denote a basic pricing policy, and $\varphi_\ell(t)$ a linear pricing policy. Let $g(\mu)$, $g_\ell(\mu)$ denote the expected payments, and $g'(\mu)$, $g_\ell'(\mu)$ denote the derivatives of the expected payments. The lesser profitability of the linear policies occurs in three ways.

If at fixed (n,μ) optimal linear and basic policies exist, it may be that $g_\ell(\mu) > g(\mu)$, and the loss to the consumer is $g_\ell(\mu) - g(\mu)$.

Neglecting $(4.3')$,$(4.4')$, it is possible that at some (n,μ) a linear policy cannot simultaneously satisfy $(4.2')$ and $(4.5')$ while a basic policy can. This is brought out by Proposition 5. One measure of the extra cost to the consumer in this event is the additional sampling required to increase (n,μ) to (n',μ), a point at which $(4.2')$ and $(4.5')$ can be simultaneously satisfied.

It is also apparent that even if a linear policy does satisfy $(4.2')$ and $(4.5')$ at (n,μ), constraints $(4.3')$, $(4.4')$ may be active, and may prevent the existence of a linear policy. An increase in sample size also measures the loss in this case. In the remainder of this section we derive statements indicating the amount of extra sampling required for the existence of optimal linear policies as a function of the negotiated parameters.

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<u>Proposition 5</u>

Recall that by $\underline{g}(\cdot)$, $\overline{g}(\cdot)$ we denote minimum and maximum expected payments. Let $t_\alpha < \mu$ and $t_\beta < t_\alpha$. If $a \leq t_\alpha$ and $b \geq a$ then $\underline{g}'(\mu) \leq \underline{g}'_\ell(\mu)$, $\overline{g}'(\mu) \geq \overline{g}'_\ell(\mu)$, and by definition of $g'(\mu)$ if more than one $\varphi_\ell(t)$ is feasible then $\underline{g}'_\ell(\mu) < \overline{g}'_\ell(\mu)$.

Proof:

The maximum value of $g'(\mu)$ is

$$(4.80) \quad \overline{g}'(\mu) = m \int_{t_1}^{t_\alpha} r(t|\mu)\, dt + w \int_{t_\alpha}^{\mu} r(t|\mu)\, dt + M \int_{\mu}^{t_2} r(t|\mu)\, dt .$$

At that (a,b) which maximizes $g'_\ell(\mu)$

$$(4.81) \quad \overline{g}'_\ell(\mu) = \overline{g}'(\mu) + \int_a^{t_\alpha} s(t-a)\, r(t|\mu)\, dt$$

$$+ \int_{t_\alpha}^{\mu} [\varphi(t_\alpha) - w + s(t-t_\alpha)]\, r(t|\mu)\, dt$$

$$+ \int_{\mu}^{b} [M - \varphi(\mu) + s(t-\mu)][- r(t|\mu)]\, dt .$$

Since $s > 0$ and $r(t) \leq 0$ for $t < \mu$ the first integral is non-positive. By $(4.3')$ $\varphi(t_\alpha) \geq w$ and the second integral is non-positive. By $(4.5')$ $\varphi(\mu) \leq M$ and the third integral is non-positive, yielding

$$(4.82) \quad \overline{g}'_\ell(\mu) \leq \overline{g}'(\mu) .$$

Similarly

$$(4.83) \qquad \underline{g}'(\mu) = v \int_{t_2}^{t_\beta} r(t|\mu) \, dt + w \int_{t_\beta}^{t_2} r(t|\mu) \, dt \ .$$

If $a = b = t_\beta$, $\underline{g}_\ell'(\mu) = \underline{g}'(\mu)$. If a is constrained to be less than b, let $s = \dfrac{w - m}{b - a}$ and consider the case in which $b \leq \mu$ and $\varphi(b) = w$. For that $a \leq t_\beta$ at which $\underline{g}_\ell'(\mu)$ is minimized we have

$$(4.84) \qquad \underline{g}_\ell'(\mu) = \underline{g}'(\mu) + \int_a^{t_\beta} [m - v + s(t-a)] \, r(t|\mu) \, dt$$

$$+ \int_{t_\beta}^b [\varphi(t_\beta) - w + s(t-t_\beta)] \, r(t|\mu) \, dt \ .$$

By $(4.4')$ $\varphi(t_\beta) \leq v$ and $m + s(t_\beta - a) = \varphi(t_\beta)$. Thus the first integral is non-negative. Also $\varphi(t_\beta) + s(b - t_\beta) = \varphi(b) = w$ and the second integral is non-negative, yielding

$$(4.85) \qquad\qquad\qquad \underline{g}'(\mu) \geq \underline{g}_\ell'(\mu)$$

and completing the proof.

We will now assume that $\underline{g}_\ell'(\mu) \leq h'(\mu)$, that $m_o = m$ and $M_o = M$ in (4.79), that $t = \dfrac{1}{n} \sum_1^n \tau_i$, and that t is normally distributed with mean μ and variance $\dfrac{1}{n} \sigma_\tau^2$. Under these assumptions we derive the minimum sample size, $\underline{n}$, at which a linear policy satisfying $(4.3')$–$(4.5')$ can yield equality in $(4.2')$. We then compare $\underline{n}$ with the minimum sample size required by an optimal basic policy. From (4.79)

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$$(4.86) \qquad g_{\boldsymbol{\ell}}(\mu) = \int_{-\infty}^{a} mf(t|\mu) \ dt + \int_{a}^{b} (m+s(t-a)) \ f(t|\mu) \ dt$$

$$+ \int_{b}^{\infty} Mf(t|\mu) \ dt \ ,$$

or

$$(4.87) \quad g_{\boldsymbol{\ell}}(\mu) = M + s(a-\mu) \ F(a) - s(b-\mu) \ F(b) - s\sigma_t[f(b) - f(a)] \ .$$

Differentiating with respect to μ we have

$$(4.88) \qquad\qquad g_{\boldsymbol{\ell}}'(\mu) = s[F(b) - F(a)] \ .$$

We seek (a^*, b^*) to maximize (4.88). While it appears as if both s and either a or b are independent, a moment's reflection shows that if two policies have identical s, and both satisfy $\varphi(t_\alpha) \geq w$, that policy with the greater a will have the greater $g_{\boldsymbol{\ell}}'(\mu)$. Therefore $\varphi(t_\alpha) = w$ and thus

$$(4.89) \qquad\qquad a = t_\alpha + \frac{m - w}{s} \ ,$$

$$(4.90) \qquad\qquad b = t_\alpha + \frac{M - w}{s} \ ,$$

demonstrating that (4.88) is a function of only one variable. If s^* maximizes (4.88) we note that at the minimum sample size, $\underline{n}$, for which $g_{\boldsymbol{\ell}}'(\mu) = h'(\mu)$, we must have $g_{\boldsymbol{\ell}}'(\mu) = \overline{g}_{\boldsymbol{\ell}}'(\mu)$, and from (4.88) we have

$$(4.91) \qquad\qquad s^*[F(b) - F(a)] = h'(\mu_o) \ .$$

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Let $u_a = \dfrac{a - \mu}{\sigma_t}$, $u_b = \dfrac{b - \mu}{\sigma_t}$, and $u_\alpha = \dfrac{t_\alpha - \mu}{\sigma_t}$. After rearrangement (4.91) becomes

$$(4.92) \qquad \frac{\sqrt{n}}{u_\alpha - u_a} \left[F_{N*}(r_1 u_\alpha - r_2 u_a) - F_{N*}(u_a) \right] = \frac{h'(\mu)\, \sigma_\tau}{w - m}$$

where $r_1 = \dfrac{M - m}{w - m}$, $r_2 = r_1 - 1$, and the subscript $N*$ denotes the standardized normal distribution. With only a table of the cumulative normal density function we can solve (4.92) for both the minimum n, $\underline{n}$, and that a which yield equality in (4.2') for a policy satisfying (4.3') and (4.5').

We derive approximations to $\underline{n}$ in terms of the parameters. By letting $a = b = t_\alpha$ in (4.86) we obtain

$$(4.93) \qquad g_\ell'(\mu) = \frac{\sqrt{n}}{\sigma_\tau} (M-m)\, f_{N*}(u_\alpha) ,$$

and since we seek the minimum n for which $g_\ell'(\mu) \geq h'(\mu)$, (4.93) yields

$$(4.94) \qquad \underline{n}_1 \geq \left[\frac{\sigma_\tau h'(\mu)}{(M-m)\, f_{N*}(u_\alpha)} \right]^2 ,$$

an upper bound to $\underline{n}$. The bound approaches $\underline{n}$ as w approaches M. We now obtain an approximation to $\underline{n}$ that is more accurate as w approaches m. By setting $b = \mu$ in (4.88),

$$(4.95) \qquad g_\ell'(\mu) = \left(\frac{M - w}{\mu - t_\alpha} \right) \left[\frac{1}{2} - F_{N*} \left(u_\alpha \left[\frac{M - m}{M - w} \right] \right) \right] .$$

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From this we derive

$$(4.96) \qquad n_2' \geq \left[\frac{u_\alpha \sigma_\tau h'(\mu)}{(w-M)\left[\frac{1}{2} - F_{N*}\left(u_\alpha\left[\frac{M - m}{M - w}\right]\right)\right]} \right]^2 .$$

The $\underline{n}$ determined by (4.92) is constrained by (4.4'), $\varphi(t_\beta) \leq v$. Thus at both $\underline{n}$ and $\underline{n}_1$ (4.2')-(4.5') hold, and we modify n_2' so that the linear policy obtained by setting $b = \mu$ also satisfies (4.4').

The statement $\varphi(t_\beta) \leq v$ implies $m + (t_\beta - a)\left|\frac{w - m}{t_\alpha - a}\right| \leq v$, from which

$$(4.97) \qquad n_2'' \geq \frac{\sigma_\tau^2}{(\mu_o - \mu_b)^2} \left[u_\beta - u_\alpha \left(\frac{M - v}{M - w}\right) \right]^2 .$$

Let $\underline{n}_2 = \max(n_2', n_2'')$. This is an upper bound on $\underline{n}$, and therefore for given parameters we choose $\min(\underline{n}_1, \underline{n}_2)$ as the upper bound on $\underline{n}$.

One lower bound for $\underline{n}$ is given by n_3, the minimum sample size required for the existence of a basic policy satisfying (4.2')-(4.5'). From (4.80) $\overline{g}'(\mu) = \frac{\sqrt{n}}{\sigma_\tau} \left[(M-w) f_{N*}(0) + (w-m) f_{N*}(u_\alpha)\right]$, and since we seek $\overline{g}'(\mu) = h'(\mu)$,

$$(4.98) \qquad n_3 \geq \left[\frac{\sigma_\tau h'(\mu)}{(M-w) f_{N*}(0) + (w-m) f_{N*}(u_\alpha)} \right]^2 .$$

Finally the minimum sample size for which $t_\beta < t_\alpha$ is given by

$$(4.99) \qquad n_4 \geq \sigma_\tau^2 \left[\frac{u_\beta - u_\alpha}{\mu_o - \mu_b} \right]^2 .$$

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The maximum of n_3, n_4 furnishes a lower bound on $\underline{n}$.

For convenience we collect the ratios $\dfrac{n_1}{n_3}$, $\dfrac{n_2'}{n_3}$, $\dfrac{n_2''}{n_4}$ below. The ratio $\dfrac{n_2''}{n_4}$ is an approximate measure of the extra sampling required by a linear policy to satisfy the price discrimination required by (4.3) and (4.4).

$$(4.100) \qquad \frac{n_1}{n_3} = \left[\frac{k f_{N*}(0) + f_{N*}(u_\alpha)}{(k+1)\, f_{N*}(u_\alpha)} \right]^2 , \qquad k = \frac{M - w}{w - m} .$$

$$(4.101) \qquad \frac{n_2'}{n_3} = \left[\frac{u_\alpha [k f_{N*}(0) + f_{N*}(u_\alpha)]}{k \left[\frac{1}{2} - F_{N*}\left(u_\alpha \frac{k+1}{k} \right) \right]} \right]^2 , \qquad k = \frac{M - w}{w - m} .$$

$$(4.102) \qquad \frac{n_2''}{n_4} = \left[\frac{u_\beta - u_\alpha r}{u_\beta - u_\alpha} \right]^2 , \qquad r = \frac{M - v}{M - w} .$$

Figure 1 displays the minimum of $\dfrac{n_1}{n_3}$, $\dfrac{n_2'}{n_3}$ as a function of $k = \dfrac{M - w}{w - m}$ and α. This indicates approximately the degree of extra sampling required for linear policies to provide the same motivation as do basic policies. The figure overstates the amount of extra sampling required in all cases.

Thus under the assumptions that t is a sample mean, normally distributed with known variance, and that the linear policy is obtained by negotiating the maximum and minimum payments and the points a, b, at which these payments occur, we can calculate from (4.92) the minimum sample size required for the existence of an optimal linear pricing policy. Comparing this to the sample size required for the existence
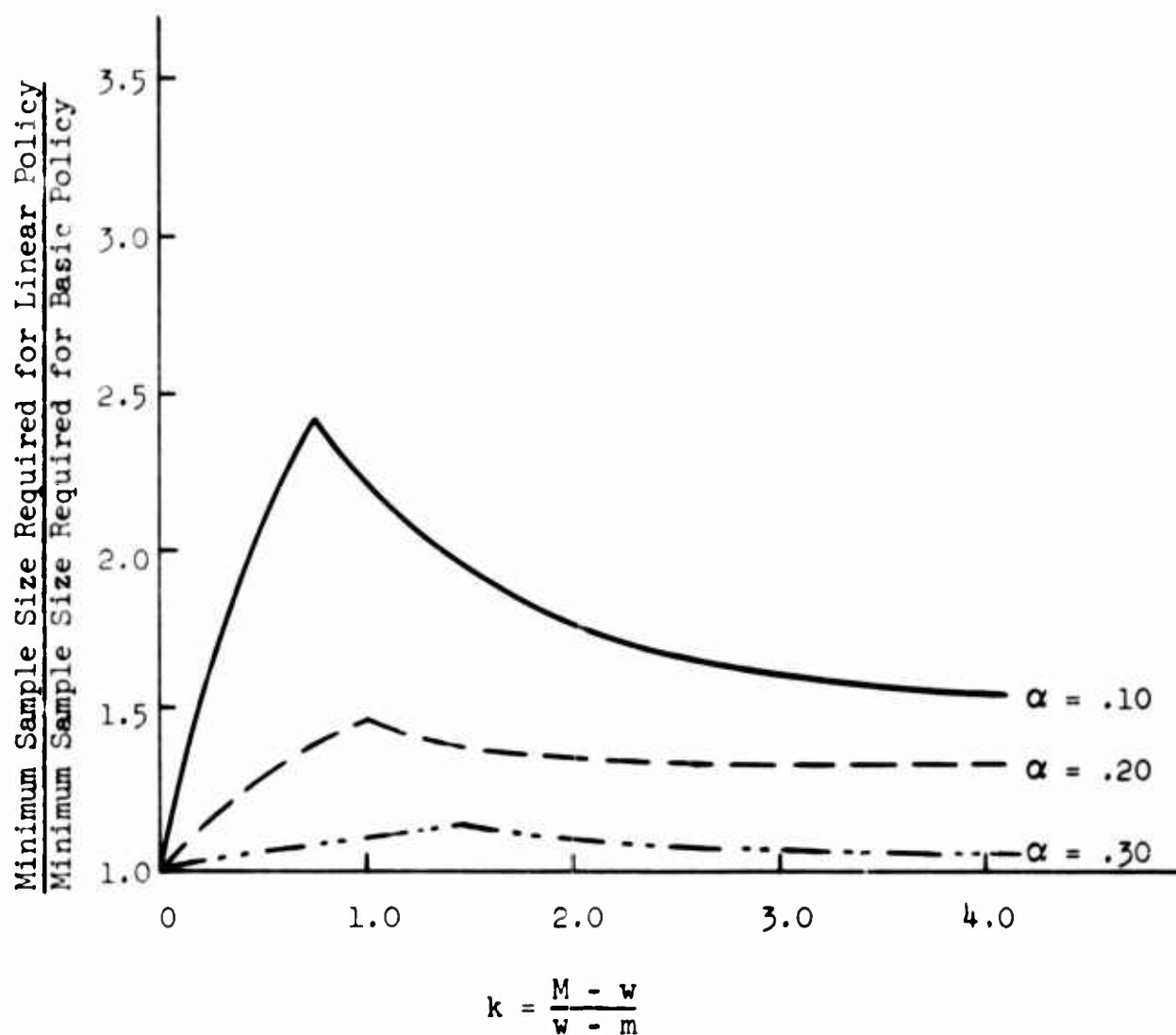
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$$k = \frac{M - w}{w - m}$$

Figure 1

of an optimal basic pricing policy provides one measure of the cost to the consumer of using optimal linear policies. Figure 1 indicates that we will not be very inaccurate by stating that linear policies will generally require from 1.1 to 1.6 times the sample size required by an optimal basic policy.

If linear policies are negotiated without consumer knowledge of the sampling distribution and production cost function, the expected producer profit will not in general be maximized at the desired performance level, μ_o. Let a, b, be selected symmetrically about μ_o, and let

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M, m, be selected symmetrically about $h(\mu_o)$. If the sampling distribution is symmetric about μ_o, then $g(\mu_o) = h(\mu_o)$. If the probability of t falling outside [a, b] is negligible then $\iota'(\mu) \approx m + \frac{M-m}{b-a} \mu$, and by selecting $\frac{M-m}{b-a} = h'(\mu_o)$, motivation at μ_o is possible. But if the probability of t falling outside the incentive region is not negligible, then $g(\mu) < m + h'(\mu_o) \mu$ for $\mu > \mu_o$ and $g(\mu) > m + h'(\mu_o) \mu$ for $\mu < \mu_o$, leading to a much weaker profit motivation in the case of convex production costs than the linearity of $\varphi(t)$ suggests. If the consumer intends to use a piecewise linear pricing policy, he must negotiate to keep (b-a) large to compensate for lack of information describing the sampling distribution, and can reduce the sample size required to maximize expected producer profit at μ_o by keeping $M - m$ large relative to $\sigma_\tau h'(\mu_o)$.

CHAPTER V

CONCLUSIONS

We briefly review the factors leading to the contingent pricing
model developed in this paper, summarize the results obtained, and comment
on the implications of the results in negotiating or formulating contin-
gent pricing contracts. In discussing the use of contractual incentives
we noted that in addition to expected profit, a contractor or producer
considers the risk of financial loss, the maintenance of corporate
reputation, and the desire for both program continuation and future
contracts in making program decisions. We concluded that expected profit
alone is an uncertain motivator and that while the overall effect of
contractual incentives on industrial efficiency has been beneficial, the
government as consumer cannot predict the program decisions of the con-
tractor on the basis of contractual profit alone. Thus incentive arrange-
ments can be analyzed for consistency with government objectives, but
not for motivation. Of course as sharing proportions increase, the
relative weight of profit in program decisions will increase.

Following these qualitative considerations we concentrated on the
specific and idealized problem of formulating consistent incentive pay-
ment plans, or contingent pricing policies, when the payment is based
on the result of testing or inspecting a sample of the items contracted
for. We reviewed two models which deal specifically with situations of
this type, but which do not yield sufficiently realistic pricing policies.

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We suggested that more realistic policies could be derived by assuming that while the consumer and producer each attempt to negotiate a contingent pricing contract which maximizes their individual expected net gains, both require that the contract include acceptable protection against possible overpayment or underpayment due to sampling variation. We postulate a model in which p, the fraction of defective items in submitted batches, is controllable by the producer at known cost, $h(p)$, and in which payments are based on the number of defectives observed in a sample drawn from each batch. We defined an optimal pricing function as one which motivates the producer to select that p_o desired by the consumer, yields a producer expected profit at p_o no less than that agreed on, and satisfies restrictions against the risk of incorrect payment. If an optimal price schedule exists at a fixed sample size, n, and quality level, p, it can be determined by linear programming. The linear programming formulation indicates that a price schedule optimal by our definition is a piecewise constant function with the number of distinct price levels related to the number of restrictions the producer and consumer include to control the risk of incorrect payment. We present several examples of such policies in Chapter III, and refer to these piecewise constant pricing policies as "basic" policies.

We next restricted pricing policies to be linear in the number of defectives observed within some interval $[a, b]$ and constant outside that interval. We refer to such piecewise linear policies as "linear," and illustrated with an example that optimal linear policies can be derived by solving a sequence of linear programming problems. We indicated briefly how the basic model could be extended to situations in

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which production control and production cost are not precise, but can be described in terms of known probability distributions. When considering several variables, additive, independent, incentive payment schedules do not adequately reflect the consumer's valuation of alternative outcomes outside a small region. A joint pricing function of all of the relevant variables is required rather than a linear combination of independent payments. In Chapter IV we demonstrated that if the joint density function of the sample outcomes is known, joint pricing functions can be determined by linear programming, but that this seemed computationally impractical for more than two variables. A more realistic and tractable approach is to define the desired expected payment function in terms of the performance parameters of interest, and seek a pricing function of known form such that the expected value of this pricing function approximates the desired expected value function.

We modified the basic model to deal with test outcomes which are continuous variables. In such cases approximations to optimal price schedules can be computed by linear programming, but the maximum principle of control theory yields the information that the form of the optimal price function, $\varphi(t)$, is piecewise constant, and indicates the points at which the price levels in $\varphi(t)$ can shift. More important, the maximum principle enables us to explain the widespread use of linear pricing functions in terms of a natural criterion. Assume that the consumer desires the producer to produce at some quality or performance level, μ_o, and that t estimates μ. Let "excess" profit denote the difference between actual profit received by the producer and the negotiated target profit at μ_o. Subject to very weak conditions on the

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distribution of t, that pricing policy which both maximizes producer expected profit at μ_o and minimizes mean square excess profit at μ_o is piecewise linear in t. If the risk of incorrect payment due to sampling variation is not explicitly considered, this result implies that the piecewise linear policies in current use are in fact optimal under a very natural criterion.

If the consumer uses linear policies that explicitly take account of the risk of overpayment, he may incur extra costs. At some (ι,μ_o) a linear policy capable of motivating production at the desired μ_o, satisfying a profit non-negativity constraint at μ_o, and remaining within specified bounds, may not drive expected excess producer profit as low as an optimal basic policy. Letting $g_\ell(\mu)$ and $g(\mu)$ denote the expected payments of the linear and basic policies, and $h(\mu) + z$ denote the production cost at μ plus the agreed on profit at μ, the situation just described yields a resultant extra cost to the consumer of $g_\ell(\mu_o) - g(\mu_o)$. The occurrence of this situation is related to the magnitude of $(M-m)$, the difference between the maximum and minimum bounds within which the actual payment schedule is negotiated. As this difference increases, the loss given by $g_\ell(\mu_o) - g(\mu)$ decreases.

Extra cost may arise if at some fixed sample size, expected producer profit cannot be maximized at μ_o by a linear policy. Proposition 5, Chapter IV, states that a basic policy can always motivate production at a specified point, μ_o, with a smaller sample size than can a linear policy. The exact minimum sample size required for the linear policy can be computed by (4.92) for specified parameters. Figure 1, Chapter IV presents an approximate ratio of the sample sizes required

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for optimal linear and basic policies to motivate production at some μ_o. We have defined w to be that payment which the producer is assured of receiving with probability $1 - \alpha$ if his quality or performance is at least as good as μ_o. Let σ_τ^2 be the variance of a single test or inspection outcome. The sample size required for motivation at μ_o is related to the derivative of the production cost function at μ_o, $h'(\mu_o)$. We can say approximately that when τ_i is an observation and t is a sample mean, the sample size required for the existence of an optimal policy linear as a function of t is directly proportional to $\sigma_\tau h'(\mu_o)$, and inversely proportional to both $(M-w)$ and $(M-m)$. Therefore it behooves the consumer to negotiate for large $(M-m)$ and the lowest price level w to which the producer will agree.

If the consumer insists that he will pay no more than v with probability $1 - \beta$ when true quality is no better than μ_b, a quality level designated as poor, and the producer insists on a high degree of assurance that he will receive at least w for appropriate performance, the linear policy will not provide the required price discrimination at as small a sample size as the basic policy. An approximate ratio of the sample sizes required for the linear and basic policies due to the need for price discrimination is

$$
(5.1) \qquad \frac{\text{sample size required for linear}}{\text{sample size required for basic}} = \left[\frac{u_\beta - \left(\frac{M-v}{M-w} \right) u_\alpha}{u_\beta - u_\alpha} \right]^2
$$

where u_β and u_α are the $1 - \beta$ and α percentage points of the cumulative standardized normal density function. From (5.1) we see that the sample size required for the existence of an optimal linear policy

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increases as the degree of price discrimination required in the incentive payment schedule increases. When price discrimination is not explicitly considered, the sample size depends only on $(M-m)$, σ_τ, $h'(\mu_0)$, and the confidence with which the consumer desires to distinguish between alternative performance levels.

Recall that a and b are the endpoints of the linear section of the policy $\varphi(t)$. Assume the sampling distribution is symmetric about its mean, and that a, b are selected symmetrically about the sample outcome estimating the desired performance level μ_0. If $(b-a)$ is small relative to σ_τ the linear portion of the policy does not apply over much of the region in which the sample outcomes are expected. The result is much weaker producer profit motivation toward high performance, and less financial deterrence from low performance than the linearity of $\varphi(t)$ suggests. When sampling distributions are not known precisely the consumer must attempt to retain linearity over a wide region, by keeping $(b-a)$ large, in order to have any estimate of the profit motivation acting on the producer.

We conclude that piecewise linear pricing policies can yield an expected producer profit which is maximized at the performance level desired by the consumer, and in fact are optimal policies under one criterion. While linear policies are readily interpreted by the producer and have the desirable practical property of continuity, their use may require larger sample sizes and consequent extra cost. Moreover, at a fixed sample size producer expected profit resulting from a piecewise linear policy does not decrease as sharply for performance below μ_0 as does the expected profit resulting from a basic policy.

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Therefore if a basic policy is unacceptable to a producer, we recommend that the linear policy negotiated include more than one linear section, with the slopes of successive linear sections decreasing. The object, of course, is to create a concave expected payment function in the neighborhood of the desired production point, thereby decreasing the financial attractiveness of performance below that desired by the consumer.

Throughout the paper we have assumed the consumer can state product value as a function of the relevant product characteristics and production or development schedule. Such knowledge does not customarily exis in functional form, and the derivation of optimal contingent pricing policies does not depend on this assumption. The essential requirements for the derivation of contingent pricing policies are knowledge of the sampling distribution and knowledge of the production cost function in the neighborhood of the performance or quality level desired. The consumer may of course derive contingent pricing policies at a variety of sample sizes and performance levels, and then select the preferred policy on economic and other grounds.

Contingent pricing policies can be implemented with little problem since producers have long been familiar with acceptance sampling, and are becoming increasingly familiar with incentive contracts. Where existing procurement contracts call for acceptance sampling, the AQL, producer's risk, LTPD, and consumer's risk can be inferred from the sampling plan in use. Starting with the requirement that the "w" of this paper be the price currently paid when batches are accepted, the contingent pricing policy can readily be computed by the methods of

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Chapter III. In new procurement actions where no acceptance sampling

arrangements exist, the problems involved in negotiating the parameters

of a contingent pricing policy do not appear more intractable than those

required to determine the proper payment and acceptance sampling plan.

The only additional essential item of information required is the pro-

duction cost function in the neighborhood of the desired AQL, and this

does not seem to be a sufficiently damaging disclosure to preclude

negotiation of such contracts.

Furthermore there may be no a priori reason to prefer one price

level w or v to another. Therefore while M and m are to be

negotiated, the sample size n can be determined by the specification

of μ_o, μ_b, and the probabilities of erroneous statements about the

parameters. The prices w and v are then determined by $\varphi(t)$. This

has the effect of creating a more symmetric policy about μ_o.

As we remarked in Chapter II, contingent pricing policies provide

attractions to both consumer and producer. When the consumer cannot

reject items due to urgent operational need or planned production require-

ments, a single acceptance price does not provide a sufficiently powerful

tool to motivate superior producer effort. If the producer purchases

the same item from several producers, all may submit acceptable items,

but contingent pricing policies allow the conscientious producer to be

rewarded with an immediate increase in dollar payments rather than a

slightly greater percentage of accepted lots in the long run. Lots

previously rejected by the consumer and scrapped may in some instances

be accepted at reduced cost, thereby averting a total loss for the

producer.

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The approach developed in this paper can be used to test existing or proposed incentives for consistency with consumer objectives, and to formulate optimal contingent pricing policies in situations where true performance must be inferred from sample data.

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